Optimal Immunization Strategy in Multiple Period Portfolio Selection

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摘要

在金融市場上，由于不同的利率劇烈變化，擁有大量資產與負債的金融機構都會採用久期加負債利率風險管理的數學模型，調整債券投資組合，以期達到正淨資產的目標。但是，傳統的久期加負債利率風險管理並不完善，所以文獻中出現了很多有關的改良方法。本論文試圖利用動態規劃改善現存方法中四個不完善之處。

本文分爲兩個部份，首部分將會介紹基本的金融知識 - 債券和市場收益率，並介紹有關久期加負債利率風險管理的文獻。第二部份將會提出一個全新的久期加負債利率風險管理方法 - 久期加負債利率風險動態債券投資方法。

Abstract

Immunization is a method of investing available funds in order to immunize risks in meeting future obligations. A number of existing immunization methods and modifications in the literature, such as duration-matching method and duration targeting method, have been critized for their methodological shortcomings. In particular, immunization has been criticized as a myopic, inaccurate description of reality. This thesis will introduce a new method of immunization. This thesis does not solve all such shortcomings, suffered by the duration-matching method and the duration targeting method, but rather concentrates on a few in particular. In short, it will focus on immunizing risks in meeting obligations dynamically.

This thesis is divided into four chapters. Basic financial knowledge - especially concerning the nature of bonds and yields - will be first introduced, followed by a review of the literature apropos to extant immunization methods. The later part of this thesis will mainly discuss a new immunization method - the optimal immunization strategy in multi-period portfolio selection and its numerical implementation.

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Preface

Many corporates such as a mutual pension fund company, who is facing a series of cash obligations, would always wish to acquire a portfolio (formed from a set of N securities) that will pay these obligations as they arise.

One way to approach this problem is to purchase a set of zero-coupon bonds that have maturities and face values exactly matching the individual obligations. However, this simple technique may not be feasible since there are few corporate zero-coupon bonds. If perfect matching is not possible, you may instead acquire a portfolio having a value equal to the present value of the stream of obligations. You can sell some of your portfolio whenever cash is needed to meet a particular obligation; or if your portfolio delivers more cash than needed at a given time (from coupon or face value payments), you can buy more bonds. If the yields do not change, the value of your portfolio will, throughout this process, continue to match the present value of the remaining obligations. Hence you will meet your obligations exactly.

A problem with this present-value-matching technique arises if the yields change. The value of your portfolio and the present value of the obligation stream will both change in response, but probably by amounts that differ from one another. Your portfolio may then not match the obligation.

Immunization(or duration-matching method) solves this problem, at least approximately, by matching duration as well as present values. If the duration of the portfolio matches that of the obligation stream, then the cash value of the portfolio and the present value of the obligation stream will respond identically (to first order) to a change in yield (but only if the change of the interest rate is small enough).

Specifically, if yields increase, the present value of the asset portfolio will decrease by approximately the same amount as the present value of the liability; so the value of the portfolio will still be adequate to cover the obligation.

There are many modifications to this duration-matching method within the available literature. However, most of these variations consider only one interest rate change, while the interest rate change itself is assumed to be very small; concurrently, they assume the liability rate and the spot rate to be the same.

Therefore, there exist a number of unfavorable constraints that make immunization extraneous in reality. In practice, liability rates and the spot rates are usually different, with these rates possibly changing more than once within a time horizon under consideration. Moreover, changes in the interest rate may not be so minuscule. In our research, we are trying to find a rule for portfolio selection, which can immunize obligations with assets. The general purpose of our research is to do immunization dynamically.

Chapter 1 describes some basic financial knowledge, especially concern the bond's nature and yield's nature. Also, some specific terms (i.e. Duration, Convexity and Time Value) will be introduced.

Chapter 2 describes extant immunization methods focusing upon the laddered strategy, the dumbbell strategy, the immunization strategy and the duration targeting strategy etc.

Chapter 3 propose a new optimal immunization strategy in multi-period portfolio selection.

Chapter 4 is the summary of the implementation of the portfolio selection method described in the previous chapter. Also, future works about the new immunization method will be pointed out to discuss too.

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Chapter 1

Background

1.1 Bond and Yield

1.1.1 Bond [8]

A bond is an obligation by the bond issuer to pay money to the bond holder according to rules specified when the bond is issued. Generally, a bond pays a specific amount, usually its face value(par value) at the date of maturity; some bonds are even accompanied by periodic coupon payments. Bonds generally have par values of even amounts and the coupon payments are often described as a percentage of the face value.

In the past, actual coupons were attached to the bond certificates themselves. The bond holder would mail these to the agent of the issuer (usually a bank) one at a time, at specified dates, and the appropriate coupon payment would then be sent by return mail. These physical coupons are rare today, but the name "coupon" remains.

Maturity date of a bond is the date when the bond mature. At the maturity date, the last coupon, together with the face value will be paid by the bond issuer.

The coupon amount is described as a percentage of the face value. For example, a 10% coupon bond with a face value of \$100 will have a coupon of \$10 per year. However, the period between coupons may be less than a year. In some places, coupon payments are made every 6 months, paying one-half of the coupon amount. Thus, a 10% coupon bond with a face value of \$100 will have a coupon of \$5 half year in this case.

Some coporations will sell bonds to raise capital immediately, (these corporations are the issuers of the bonds), and hence forth obligated to make the prescribed payments. Usually bonds are issued with coupon rates close to the prevailing general rate of interest so that they will sell at a price which is close to their face value. For example, when the spot rates are r_i at time i, the price (P) of a $c\%$ coupon bond who pays coupons at time $1, 2, 3, \ldots, T-1, T$, with face value F matures at time T is:

$$P = \sum_{k=1}^{T-1} \frac{F \times c\%}{(1+r_k)^k} + \frac{F \times (1+c\%)}{(1+r_T)^T} \tag{1.1}$$

However, as time passes, bonds are no longer traded at a price determined by the interest rates, instead, prices are determined by the market. Any two parties can agree on a price and execute a trade; the vast majority of bonds are sold either at auction (when originally issued) or through an exchange organization. The price is therefore determined by a market and thus may vary time by time.

1.1.2 Yields

The word yield is frequently used with regard to investing in bonds. There are two important types of yields that the investor must be familiar with: the current yield and the yield to maturity. This section will differentiate among these two yields.

The Current Yield

The current yield is the percentage that the investor earns annually.

$$\frac{Annual\ Interest\ Payment}{Price\ of\ the\ Bond}$$

For example, when the price of a 10% coupon bond of face value \$100 is sold at a price of \$95.2, the current yield will be:

$$\frac{\$10}{\$95.2} = 10.504\%$$

The current yield is important because it is an indicator of the current return that will be earned on the investment. Investors who seek high current income prefer bonds that offer a higher current yield.

However, the current yield can be very misleading for it fails to consider any change in the price of the bond that may occur if the bond is held to maturity. Obviously, if a bond is bought at a discount, its value must rise as it approaches maturity. The opposite occurs if the bond is purchased for a premium, because its price will decline as maturity approaches. For this reason it is desirable to know the bond's other yield - yield to maturity.

The Yield to Maturity

A bond's yield is the interest rate implied by the payment structure. Specifically, it is the interest rate at which the present value of the stream of payments (consisting of the coupon payments and the final face-value payment) is exactly equal to the current price, in other words, it is just the internal rate of return of the bond at the current price. This value is termed more properly the yield to maturity -YTM (instead of the internal rate of return) when discussing bonds.

Let F be the face value of a bond, m be the number of coupon payments each year, C be the coupon rate, n be the number of years remaining and i be the yield to maturity of a bond (quoted at annual basis). Then the current price (P) of the bond would be:

$$P = \frac{F}{[1 + (i/m)]^n} + \sum_{k=1}^{n} \frac{C/m}{[1 + (i/m)]^k} \tag{1.2}$$

1.1.3 Qualitative Nature of Price-Yield Curves

As the bond equation is quite complex, it is easier to obtain a qualitative understanding of the relationship between price, yield, coupon, and time to maturity. Such a qualitative understanding, and its subsequent insights into interest rate risk properties of bonds, are essential to bond portfolio construction.

As a general rule, the yields of various bonds clearly track one another and the prevailing interest rates of other fixed-income securities. For instance, most people would not buy a bond with a yield of 6% when a bank is offering 10% saving rate. The general interest rate environment exerts a force on every bond, urging its yield to conform to that of other bonds.

However, the only way that the yield of a bond can change is for the bond's price to change. So as yields move, prices move correspondingly. But the price change required to match a yield change varies with the structure of the bond (its coupon rate and its maturity). So as the yields of various bonds move more or less in harmony, their prices move by different amounts. To understand bonds, it is important to understand this correlation between the price and yield. For a given bond, this relationship is shown periodically by the price-yield curve.

Examples of price-yield curves are shown in 1.1. Note that 15% bond means a bond who pays 15% of the face value as coupon per year.The first obvious feature of the curve is that it has negative slope; that is, price and yield have an inverse relation. If yield goes up, price goes down. If I am to obtain a higher yield on a fixed stream of recieved payments, the price I

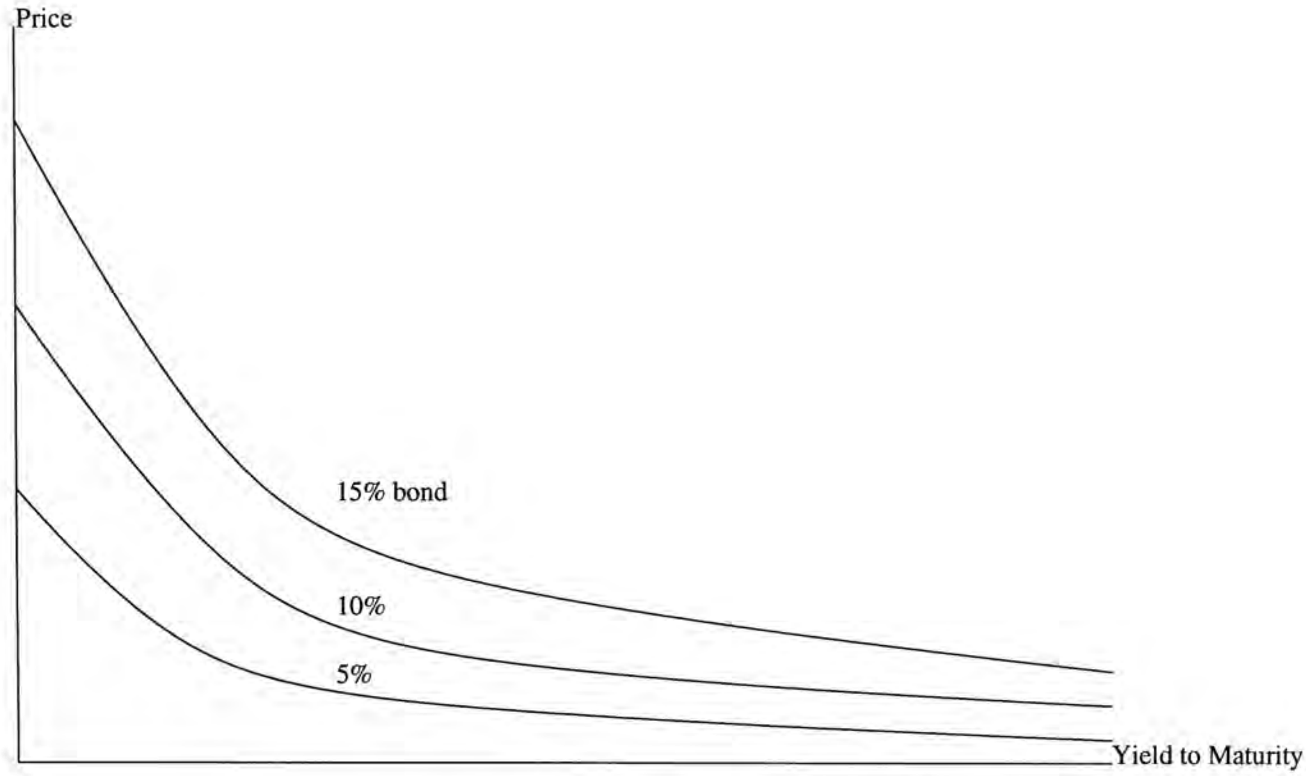

Figure 1.1: Price-Yield cuves and Coupon rate.

pay for this stream must be lower. This is a fundamental feature of bond markets. When people say - the bond market went down, they mean that the interest rates went up.

The price-yield curve is important because it describes the interest rate risk associated with a bond.

Some points on the curve can be calculated by inspection. First, suppose that YTM=0. This means that the bond is priced as if it offered no interest. Within the framework of this bond, money in the future is not discounted. In that case, the present value of the bond is just equal to the sum of all payments, it is actually the value of the bond at zero yield. Second, suppose that $YTM = 10\%$ of maturity, then the value of the bond is equal to the par value. The reason for this is that each year the coupon payment just equals the 10% yield expected on the investment. The bond is like a loan where the interest on the principal is paid each year and hence the principal remains constant. In this situation, where the yield is exactly equal to the coupon rate, the bond is termed a par bond. In addition to these two specific points on the price-yield curve, we can deduce that the price of the bond must tend toward zero as the yield increases. Overall, the shape of the curve is convex since it bends toward the origin and out toward the horizon axis. Just given these particular two points and a rough knowledge of its shape, it is possible to sketch a resonable approximation to the curve.

Bond holders are subjected to yield risk in the sense described: if yields change, bonds' prices also change accordingly. This is an immediate risk, affecting the near-term value of the bond. You may, of course, continue to hold the bond and thereby continue to recieve the promise coupon payments and the face value at maturity. This cash flow stream is not affected by interest rates. (That is after all why the bond is classifed as a fixed -income security.) But if you plan to sell the bond before maturity, the price will be governed by the price-yield curve.

	Yield				
Time to Maturity	5%	8%	9%	10%	15%
1 Year	\$104.76	\$101.85	\$100.91	\$100.00	\$95.65
5 Years	\$121.64	\$107.98	\$103.89	\$100.00	\$83.24
10 Years	\$138.61	\$113.42	\$106.42	\$100.00	\$74.91
20 Years	\$162.31	\$119.63	\$109.13	\$100.00	\$68.70
30 Years	\$176.86	\$122.52	\$110.27	\$100.00	\$67.17

Table 1.1: Prices of 10% Coupon Bonds (Face Value=\$100)

Table 1.1 displays the price-yield relation in tabulor form for bonds with a 10% coupon rate. It is easy to see that the bond with 30-year maturity is much more sensitive to yield changes than the bond with 1-year maturity.

1.2 Duration, Convexity and Time Value

1.2.1 Duration

Everything else being equal, bonds with longer maturities have steeper price-yield curves than bonds with shorter maturities. Hence, the prices of long-term bonds are more sensitive to interest rate changes than those of short-term bonds. This is shown clearly in Table 1.1. However, maturity itself does not give a complete quantitative measure of interest rate sensitivity. Instead, another measure of time length termed **duration** does give a direct measure of interest rate sensitivity. The duration of a fixed income instrument is a weighted average of the times that payments are made. The weighting coefficients are the present values of the individual cash flows.

We can write out this definition more explicitly. Suppose that cash flows are recieved at times $t_0, t_1, t_2, t_3, ..., t_n$. Then the duration of this stream is

$$D = \sum_{k=1}^{n} \frac{PV(t_k)t_0}{PV} \tag{1.3}$$

In this formula the expression $PV(t_k)$ denotes the present value of the cash flow that occurs at time t_k. The term PV in the denominator is the total present value, which is the sum of the individual $PV(t_k)$ values.

Note that the expression for D is actually a weighted average of the cash flow times, thus it has a unit of time.

Clearly, a zero-coupon bond, which makes only a final payment at maturity, has a duration equal to its maturity date. Non-zero coupon bonds have durations strictly less than their maturity dates. This shows that duration can be viewed as a generalized maturity measure. It is an average of the maturities of all the individual payments.

Macaulay Duration

To make the preceding definition of the term - duration clearer, especially about how the present value is calculated; that is, what interest rate to use. For a bond it is natural to base those calculations on the bond's yield. If the yield is used, the general duration formula becomes the Macaulay Duration.

Suppose the following: a bond pays m coupons per year, with the payment in period k being c_k , and there are n periods remaining. Making the expression $PV(t_k)$ in (1.3) to be in term bond's yield, we have,

$$PV(t_k) = \frac{c_k}{[1+(i/m)]^k} \tag{1.4}$$

where i is the yield to maturity and

$$PV = \sum_{k=1}^{n} \frac{c_k}{[1+(i/m)]^k} \tag{1.5}$$

Then, we would have the **Macaulay duration** D equals:

$$D = \frac{\sum_{k=1}^{n}(k/m)c_k/[1+(i/m)]^k}{PV} \tag{1.6}$$

1.2.2 Qualitative Properties of Duration

As mentioned in the previous section, the duration of the coupon bonds are always less than its maturity. It seems that there exists a directly proportional relationship between the duration and the maturity date of the bond. That is, as the time to maturity increase, the duration would increase correspondingly. However, this is not true. Let's consider the following, a 10% coupon bond which matures 50 years later with a yield equal to 5% has a duration of 17.38, if this particular bond matures 100 years later, it's duration is 20.06. If its maturity tends to forward infinite, it's duration, however, is not infinite as it should be, but rather only 20.5. In conclusion, the duration of the bond would be for sure less than the maturity date of a coupon bond. However, duration is not directly proportional to the maturity date. A very long duration might not be achieved by a bond with very long maturity; other features of a bond also help to determine a bond's length of duration

Duration and Sensitivity

One of the most important feature of duration is that it directly measures the sensitivity of a bond's price to changes in a bond's yield. This follows from a simple expression for the derivative of the present value expression.

As mentioned in the previous chapter, the present value of the cash flow k is as follow:

$$PV(t_k) = \frac{c_k}{[1 + (i/m)]^k}$$

The derivative with respect to i is

$$\begin{aligned} \frac{dPV(t_k)}{di} &= \frac{-(k/m)c_k}{[1 + (i/m)]^{k+1}} \\ &= -\frac{k/m}{1 + (i/m)} PV(t_k) \end{aligned} \tag{1.7}$$

Making use of the fact that the price is actually the total present value of the future cash flows, we have,

$$P = \sum_{k=1}^{n} PV(t_k) \tag{1.8}$$

as a consequence,

$$\begin{aligned}
\frac{dP}{di} &= \sum_{k=1}^{n} \frac{dPV(t_k)}{di} \\
&= -\sum_{k=1}^{n} \frac{(k/m)PV(t_k)}{1+(i/m)} \\
&= -\frac{1}{1+(i/m)} DP \\
&\equiv -D_M P \qquad (1.9)
\end{aligned}$$

The value D_M is named the **modified duration**. It is the usual duration modified by the extra term $-\frac{1}{1+(i/m)}$. Note that $D_M \approx D$ for large values of m or small values of i.

Note that the above equation could be re-written as

$$\frac{1}{P}\frac{dP}{di} = -D_M \qquad (1.10)$$

The left hand side is then the relative change in price (or the fractional change). Therefore, duration (D_M) measures directly the relative change in a bond's price as i changes which is actually the sensitivity of the bond price to the change of bond's yield, as mentioned.

Making use of the approximation $dP/di \approx \Delta P/\Delta i$, the equation rewritten as

$$\triangle P \approx -D_M P \Delta i \qquad (1.11)$$

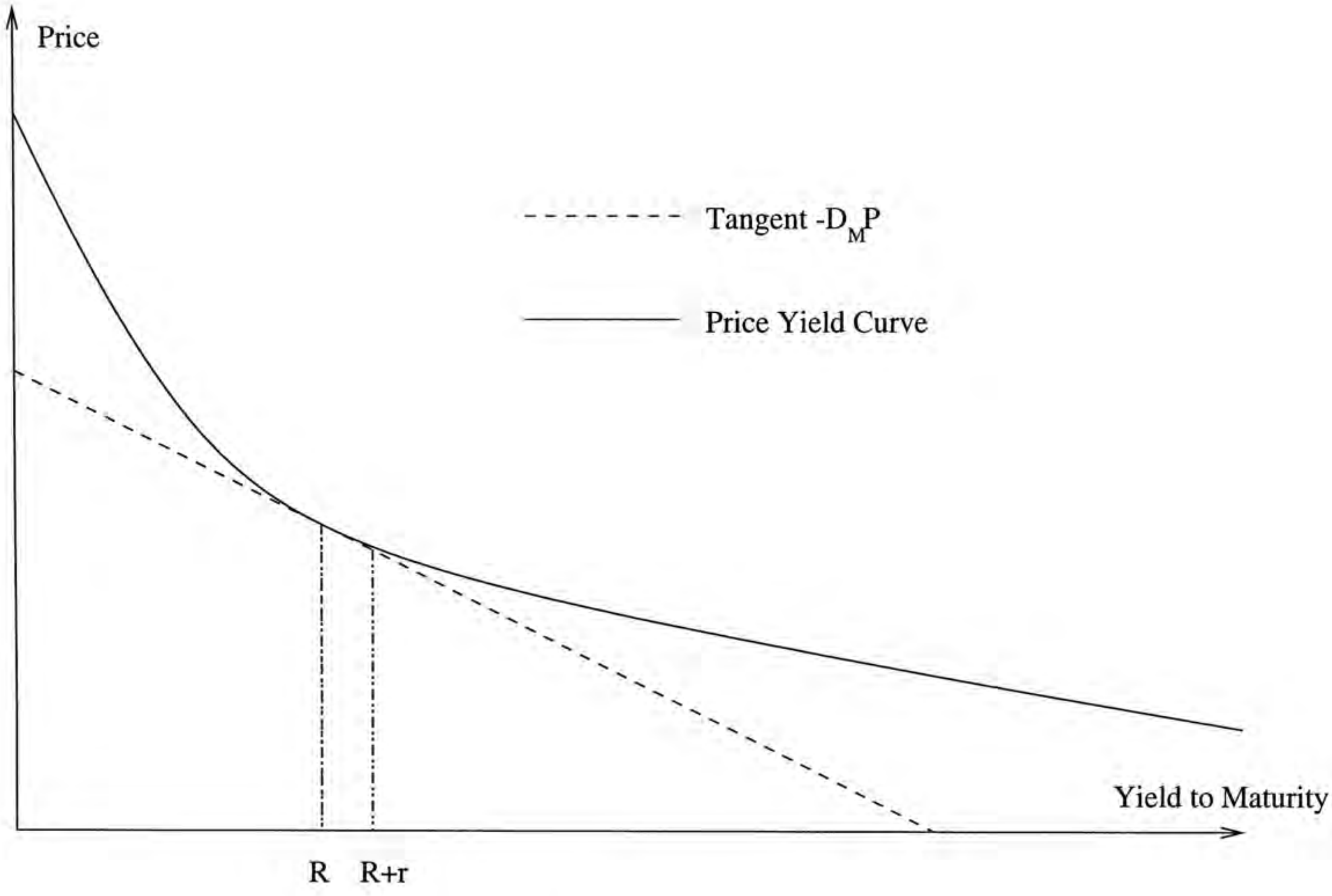

Figure 1.2: Price-yield curve and slope.

The above relation gives explicit values for the impact of yield variations.

We can see, from Figure 1.2, the price of the bond when the maturity is $R+r$ can be estimated by the straight line $-D_M P$, which is the slope of the price-yield curve at the point when yield to maturity is equal to R. This is because $-D_M P$ is a straight line approximation to the price-yield curve for nearby points.

Duration of a Portfolio

After this brief discussion of the duration of individual bonds, it is time to look at the duraiton of an entire bond portfolio. We will see, the duration of a portfolio actually measures the interest rate sensitivity of that portfolio, just as normal duration measures it for a single bond.

Suppose that a portfolio consisting of several bonds with different maturities is constructed to act like a fixed income security: it recieves periodic payments, but due to the differing maturities of the individual bonds, these payments may not be equal, though, all the bonds should have the same yield. (This is usually approximately true, since yields tend to track each other closely.) The duration of the portfolio is then just a weighted sum of the durations of the individual bonds - with weighting coefficients proportional to individual bond prices.

Specifically, for a portfolio that is made up of two bonds A and B with duraitons $D^A = \frac{\sum_{k=0}^{n} t_k PV^A(t_k)}{PV^A}$ and $D^B = \frac{\sum_{k=0}^{n} t_k PV^B(t_k)}{PV^B}$ respectively, the duration of the porfolio would be as follow,

$$D = \frac{PV^A D^A}{P} + \frac{PV^B D^B}{P} \tag{1.12}$$

where,

$$PV^A D^A + PV^B D^B = \sum_{k=0}^{n} t_k (PV^A(t_k) + PV^B(t_k))$$

This result can be easily extended to a portfolio containing several bonds - Suppose there are n fixed-income securities with prices and durations of P_i and D_i, respectively, $i = 1, 2, ..., n$, all computed at a common yield. The portfolio consisting of the aggregate of these securities has price P and duration D, given by

$$\begin{aligned} P &= P_1 + P_2 + ... + P_n \\ D &= w_1 D_1 + w_2 D_2 + ... + w_n D_n \end{aligned}$$

where $w_i = P_i/P$,$i = 1, 2, 3, ..., n$.

So far, we consider the yield of all the bonds in the portfolio to be the same , though, this is not true in reality. (Different bonds would usually have different yields.) For a portfolio which is composed by bonds with different yields, the composite duration as defined can still be used as an approximation. In this case a single yield must be chosen - perhaps the average. The present values can be calculated with respect to this single yield value, although these present values will not be exactly equal to the prices of the bonds. The weighted average duration, calculated as shown, will give the sensitivity of the overall present value to the change in the yield figure that is used.

1.2.3 Convexity

As mentioned above, modified duration measures the relative slope of the price-yield curve at a given point. As we have seen, this leads to a straight-line approximation to the price yield curve that is useful as a means of assessing risk and controlling it. An even better approximation can be obtained by including a second-order (or quadratic) term. This second-order term is based on convexity, which is the relative curvature at a given point on the price-yield curve. Duration and convexity determine the variability of prices associated with changes in the interest rate $i^A(t)$. In the case of a marginal change in the interest rate of Δi^A, the return of a bond can be determined by the expression:

$$\frac{\Delta P}{P} = \frac{\partial P}{\partial (i^A)} \frac{1}{P} \Delta i^A + \frac{1}{2} \frac{\partial^2 P}{\partial (i^A)^2} \frac{1}{P} (\Delta i^A)^2 + ... \tag{1.13}$$

There are infinite number of terms in this expression. Duration is defined as the coefficient of the first term multiple by minus one, i.e.,

$$D = -\frac{\partial P}{\partial i^A} \frac{1}{P} \tag{1.14}$$

while the convexity is the coefficient of the second term.

$$C = \frac{1}{2}\frac{\partial^2 P}{\partial (i^A)^2}\frac{1}{P} \tag{1.15}$$

The price of a bond decreases when the interest rate rises. Hence, the duration of a bond is positive. The relative rise in convexity when there is a decrease in the interest rate is greater than the relative fall in convexity when there is a corresponding increase in the interest rate.

Duration and convexity can be used to approximate changes in bond prices.

$$\frac{\Delta P}{P} \approx \frac{\partial P}{\partial (i^A)}\frac{1}{P}(\Delta i^A) + \frac{1}{2}\frac{\partial^2 P}{\partial (i^A)^2}\frac{1}{P}(\Delta i^A)^2 \tag{1.16}$$

For a small change in the interest rate, duration gives a close approximation to the actual change in the bond price. The price of a bond, however, is not a linear function of interest rate. Therefore a closer approximation is obtained if the convexity term is included. And for larger changes in the interest rate, an even closer approximation can be obtained by including the third term after the convexity term.

1.2.4 Literatures Review of Duration and Convexity

Duration and convexity, which quantify the sensitivity of bond prices to changes in interest rates, as mentioned in the previous sections, are important tools for controlling price risk caused by shifts in interest rates. In the paper of Peter Ove Christensen and Bjarne G.Sorensen [4], an additional concept called 'Time Value', defined as the bond price sensitivity toward the passing of time, was introduced. Besides, Peter [4] also shows that immunization strategies should provide not only a hedge against changes in discount rates but also a hedge against no change in discount rates. Furthermore, the paper [4] also shows that options and futures are valuable instruments for interest rate risk management.

In the financial literature, it is often assumed that the stochastic process for the term strucutre can be described by a one factor model. That is, changes in the term structure only occur as parallel shifts. The level of the term structure can change randomly over time, while the curvature of the term structure remains unchanged.

This is the reason why duration is often used for immunization of a stream of future cash flows. As mentioned before, the immunization of the future cash flows can be achieved if the portfolio is selected such that:

- the value equals the present value of the liability calculated by the current term structure of interest rates, and
- its duration equals the length of the horizon.

In the paper of Peter [4] , a portfolio and the liability having the characteristics shown in Table 1.2.4 are considered. It was shown that the value of the selected portfolio will never be smaller than the present value of the liability. The difference between the values increases with a larger change in the interest rate, irrespective of the direction. This is because the portfolio has a higher convexity than the liability.

As it was shown that it is insufficient to consider only the duration of the portfolio in immunization, therefore, the convexity of portfolio should also

Characteristic	Portfolio	Liability
Net Present Value	\$6,027,517	\$6027,517
Yield to maturity	11.03%	10.66%
Duration	5	5
Convexity	17.13	12.50

Table 1.2: Characteristics of Immunization Portfolio and Liability

be considered. Douglas suggest that maximization of the convexity of the portfolio to maximize can lead to a convexity gain due to random changes in the interest rate. However, Grantier [6] had claimed that the gain might be relatively small. In the literature of Peter [4], showed that a bond portfolio which is combined with futures and options may produce considerable gains from convexity; on the other hand, if the portfolio is constituted such that both the duration and the convexity are matched with the duration of the convexity of the liability, with interest rate changing either considerably or very little, the investor can still meet the liability at the end of the horizon. In other words, the investor has hedged against the risk of large interest rate changes as well as the risk of no changes. As a consequence, instead of maximizing the convexity of a portfolio, matching the convexity of a portfolio to that of the liability would be significantly better.

Besides the duration and the convexity, Peter [4] also explained the apparently free lunches of increasing convexity - such free lunches can be explained by the simple fact that time passes. And this leads to a dicussion of the term **time value**.

1.2.5 Time Value

The value of bond portfolios with the same current value and duration are equally affected by marginal changes in the interest rate, this follows by the feature that the duration of a particular bond actually measures the senitivity of the bond's price to the change in the interest rate. However, the portfolios may have different sensitivities to the passing of time even if the durations are matched.

The immunization argument is based on the assumption that the only thing that might happen is the instantaneous change of the interest rate. If there is no such change in the interest rate, it is assumed that the values of the portfolio and the liability change over time according to the current term structure, i.e., the term structure changes in a predictable fashion so that the future spot interest rates will be equal to the current forward rates. Hence, any two portfolios with the same current values will also have the same values at any future date.

However, Peter [4] shows that if the term structure does not change at all, the values of two portfolios will not be equally affected by the passing of time, even though the current values and durations are matched. Therefore, an effective immunization strategy must, in addition to the duration, recognize the effect of time.

Mathematically, time value is defined as the relative price change over time for a fixed interest rate,

$$TV = \frac{\partial P}{\partial t}\frac{1}{P} \tag{1.17}$$

Where TV denotes the time value, while P and t denotes the price and the time respectively. For a portfolio of bonds, the time value can be written as the value-weighted average of the forward rates for the payment dates of the bond:

$$TV = \sum_{T} f^{T}(r(t), t) w_{T} \tag{1.18}$$

Where $f^{T}(r(t), t)$ is the current forward rate for time T, and $w(T)$ is the present value of the payment of the portfolio at time T divided by the total value of the bond portfolio.

Peter [4] has shown that when assuming the non-existence of arbitrage opportunities, the larger the convexity of a portfolio, the less the value of the portfolio rises over time, if the interest rate remains unchanged.

Therefore, time effects of time can offset convexity gains, so that the net return from the immunization strategy becomes negative when there are only small changes in the interest rate. In other words, if there are not sufficiently large changes in the interest rate, the value of the portfolio at the horizon will not be large enough to meet the liability.

Chapter 2

Management of Interest Rate Risk

Since interest rates change frequently, the value of a bond portfolio fluctuates accordingly. Of course, the investor could avoid these fluctuations by acquiring only short-term debt obligations. Such a strategy generates less income since shorter maturities generally have lower yields than bonds with longer maturities. Alternatively, purchasing only very long-term bonds will increase income but also increases the risk associated with changes in interest rates. This feature was shown in the previous chapter.

In the coming sections, investment strategies that hedges against interest rate risk will be presented. In the following two sections, two traditional portfolio management methods that are believed to be myopic and unreliable but are nonetheless useful in our analysis, laddered strategy [9] and dumbbell strategy [9], would be introduced. In the third section, a bond strategy which is designed to match a portfolio with the investor's need for funds (so called immunization strategy or duration matching strategy) will also be introduced; This a bond investment strategy simply matched the present value and the duration of the bond with that of a future obligation. The fourth section briefly analyzes the importance of involving the convexity in portfolio management. In the later sections, some interest rate risk management strategies, namely duration targeting strategy and the duration vector approach, from recent literature will be presented;

2.1 Laddered Strategy

The investor could construct a portfoilio of bonds with maturities distributed over a period of time. Such a strategy is sometimes referred to as a laddered approach. For example, say you devide a \$100,000 portoflio into \$10,000 portions that mature for each of the next ten years. If interest rates change, the price of the short term bonds will fluctuate less those of the of the long term bonds. Hence, such a portfolio reduces the impact of changes in interest rates.

In addition to reducing the impact of fluctuating interest rates, a laddered portfolio offers two important advantages. First, since the structure of yields is generally positive, the interest earned from the bonds will tend to be greater than would be earned on a portfoilio of only short-term debt instruments. (Correspondingly, it would be smaller than the interest earned on a portfolio consisting solely of bonds with long terms to maturity.) Second, since some of the bonds will mature yearly, that means the liquidity of the funds are maintained; if these funds are not required, the individual may re-invest them. Should the funds be re-invested each year in bonds with a ten-year maturity, the original structure of the portfolio is retained.

One disadvantage of such a portfolio is that; if the investor anticipates a change in interest rates and wants to alter the portfolio, then virtually all bonds will have to be liquitated. If the investor anticipates lower interest rates, then a portfolio consisting of only long maturities is desirable. Alternatively, if the individual anticipates higher rates, a portfolio consisting of only short-term scurities would be ideal. In either case, all the bonds with intermediate to long terms maturities will have to be sold.

2.2 Dumbbell Strategy

In the dumbbell strategy (or barbell strategy), the investor acquires a portfolio of bonds consisting of very long-term together with very short-term maturities. If the individual has a certain amount of principal to invest, half of them should be invested in long-term bonds. If the investor then anticipates a change in interest rates, only half of the portfolio needs to be changed. An expectation of lower rates would simply require selling the short-term bonds and investing the proceeds in the long terms. If the investor anticipates higher interest rates, he or she would do the opposite: sell the long term bonds and move the proceeds into shorter-term bonds.

If the investor anticipates correctly, a dumbbell strategy will reduce the impact of fluctuating interest rates, however, the impact will be magnified if the investor is incorrect. A movement into long-term bonds just prior to an increase in interest rates could inflict a substantial loss on the value of the portfolio. This strategy also has a second major disadvantage: As time goes by, the short-term bonds will mature, and the maturities of the long-term bonds will diminish. Thus, this bond strategy requires active management, as the proceeds of the maturing bonds will have to be re-invested and some of the longer bonds may have to be sold and the proceeds invested in bonds with even longer maturities. However, if the investor keeps anticipating a lower interest rate, then the investor's cash position will increase and the term of the remaining bonds will decrease.

2.3 Immunization Strategy

The dumbbell strategy is designed to facilitate the swapping of bonds with different terms in order to capitalize from anticipated changes in interest rates. Other bond strategies may be designed to match the portfolio with the investor's need for the funds. Two of these strategies are called immunization and dedication. An immunized portfolio seeks to match the duration and the present value of the portfolio with that of the investor's obligations. The individual determines the time period over which he or she wants to lock in current yields. Then the individual constructs a portfolio of bonds with different maturities such that the duration of the portfolio is matched with the desired time period.

A dedicated bond portfolio seeks to match the receipt of cash flows with the need for funds. Hence, interest payments and principal repayments are matched with the investor's anticipated costs. For example, one may construct a portfolio of zero coupon bonds, each of which matures when his or her obligations are due.

Interest rate risk is irrelevant for both immunized and dedicated bond portfolios. By matching the duration and the present value of the portfolio with the duration and the present value of the investor's liabilities, or by timing cash receipts with cash needs, the impact of interest rate fluctuations is minimized. Such strategies are better than a simple buy and hold strategies - the laddered strategy and the dumbell strategy - because they seek to match the portfolio with the need for funds. Since a simple buy and hold does not consider when the funds will be needed, the investor will be subject to interest rate risk. The funds may possibly be needed during a period of higher interest rates, in which case the investor will not realize the value of the initial investment.

2.4 Consideration of Convexity for Managing Interest Rate Risk

We have already discussed convexity in the previous chapter, so, we will move on to discuss the advantages of we involving convexity in the portfolio mangement.

If a portfolio is constituted such that both the duration and the convexity are matched with the duration and the convexity of the liability, we obtain the following.

First, the effect of shortening the time to maturity is the same for the portfolio as for the liability. Therefore, whether the interest rate changes considerably or very little, the investor can meet the liability at the horizon. In other words, the investor has hedged the risk of large interest rate changes as well as the risk of no changes.

Second, since the duration of a portfolio always equals the liability, the need for a constant rebalancing of the portfolio will be reduced. This follows from the fact that convexity measures the marginal change in duration for a marginal change in the interest rate. When the convexity is matched, the duration of the portfolio and the liability is therefore influenced in the same way by a correspondant change in the interest rate.

Research has developed a measure of the sensitivity of an immunization strategy towards the kind of changes that occur in the term structure, the immunization risk measure M^2. If M^2 is low, then immunization will be almost complete, not only for parallel shifts in the term structure but also for more complex shifts such as twists in the term structure.

For a portfolio that immunizes against one liability, M^2 is equal to the difference between the convexities of the immunization portfolio and the liability. Therefore, the effectiveness of the immunization strategy increases as the difference between the convexities is minimized.

In general, the cash flow profile of two bond portfolios with the same duration and similar convexity will be very much alike. Hence, if both the duration and the convexity are matched, the portfolio is immunized, not only against changes in the level of interest rates, but also against random changes in the shape of the term structure of interest rates.

Therefore, it is appropriate to match duration as well as convexity if the investor wishes to hedge against a certain interest rate risk.

2.5 Duration Targeting[12]

Modern portfolio management has increasingly made use of duration targeting [12] to achieve a desired balance between risk and return. Many kinds of fixed income management are tantamount to maintaining portfoilio duration within a narrow range. Many asset allocation studies, for example, assume that the bond component is characterised by some prescribed level of volatility. Inasmuch as bond return volatility is estimated by multiplying interest rate volatility by bond portfolio duration, setting the volatility level is equivalent to setting the duration level.

Duration targeting is a strategy in which a constant duration is maintained through periodic rebalancing. One of the vital characteristics of duration-targeting strategies is the focusing of the return distribution on a minimum-variance point. This point occurs when the duration is equal to about one-half the investment horizon. Although duration targeting does not achieve a zero-return volatility, it controls investment risk reasonably well. Partial immunization of a bond portfolio can be successfully achieved by simply maintaining its duration at one-half the length of the investment horizon. This process will lead to a significant reduction in return variability at the end of the investment horizon. Because duration targeting does not achieve zero-volatility, we can view it as a form of 'partial immunization'.

Compared with duration targeting, immunization clearly offers more in the way of risk reduction at the end of the investment horizon. However, before the end of the investment horizon is reached, duration targeting offers significantly more in the way of risk reduction than immunization. Duration targeting offers greatly reduced return volatility over most of the investment horizo - but only partial, rather than full, immunization.

2.6 Immunizing Default-Free Bond Portfolios with a Duration Vector [2]

Immunization can be defined as the protection of the value of a portfolio vis-a-vis interest rate changes. It has been shown that under idealized conditions, this objective can be attained.

A standard approach to immunzation is to employ a single factor duration model analytically derived from prior beliefs regarding the nature of interest rate changes. The efficacy of this approach is compromised when actual interest rate changes are not consistent with the priors. However, Chambers and Carleton [2] have mentioned an alternative approach provided by Cooper (1977), who analyzes four simple functional forms of the term structure that express itself as a function of time:

$$R(t) = e^{\mathcal{A}+\mathcal{B}t+Clog(t)} \tag{2.1}$$

where t is a parameter representing time.

The assumption of Cooper's approach is that the term structure adheres to this particular functional form. The traditional duration approach may be viewed as a particular case of Cooper's vector approach in which the only parameter allowed to change is the height of the term structure. The paper of Chambers and Carleton [2] extend the work of Cooper's by using a less restrictive functional form of term structure. They accomplish the objective by assuming that the a polynomial may be used to fit the term structure. They also utilize a Taylor's expansion to obtain a vector of duration measures to cope with noninfinitesimal changes in the present value factors.

Chambers and Carleton[2] constructed a **Return model** for coupon bonds. The model of interest that follows below 2.2 under the assumption that there are no cash payments between time s and time $s+1$.

$$r_{i,s+1} = k + \sum_{w=1}^{\infty} D_{i,s}(w) \bullet q(w) \tag{2.2}$$

where

- $r_{i,s+1} = P_{i,s+1}/P_{i,s}$,
- $P_{i,s}$ = Price of default free bond i at time s,
- $D_{i,s}(w) = \sum_{T=1}^{\infty}(C_i(T) \bullet B_s(T)/P_{i,s})(T-1)^w$,
- T = time until promised coupon or principal payment,
- $C_i(T)$ = promised payoff of bond i in T periods,

- $B_s(T)$ =price at time s of \$1 discount bond maturing in T periods(or the discount factor at T periods.)
- k = return on single period discount bond from time s to its maturity at time $s+1$.
- $q(w)$ is a random variable that contains information regarding the term structure shift from time s to time $s+1$.

One of the features of the above model is that it incorporates general term structure behavior. This feature results in the use of a vector of duration measures $[D_{i,s}(w), w = 1, \cdots, \infty]$ rather than a scalar.

For the case of $w = 1$, the Duration vecotr $D_{i,s}(w)$ is actually identical to the traditional Fisher-Weil duration: it measures the responsiveness of bond i to a prallel shift in the term structure, where a shift in the term structure is defined as the difference between the actual term sructure at time $s+1$ and the term structure that was anticipated under the unbiased expectations hypothesis. Cases $w \geq 2$, $D_{i,s}(W)$ may be viewed as a weighted average of "time-to-maturity minus one" raised to the power w. Thus, $D_{i,s}(2)$ is a value weighted average of the "time-to-maturity squared" of all of the bond's promised cash flows. An approximate understanding of $D_{i,s}(2)$ is that is measures the responsiveness of bond i to a slope change in the term structure. Cases of $w \geq 3$ may be associated with curvatures shifts on the term structure.

Theoretical portfolio return equation [2] analogous to equation(2.2) is formed as below:

$$r_{p,s+1} = k + \sum_{w=1}^{\infty} \sum_{i} y_i \bullet D_{i,s}(w) \bullet q(w) \tag{2.3}$$

where

- $r_{p,s+1}$ =portfolio return from time s to time $s+1$,

- y_i =proportion of portfolio in bond i.

Essentially, the value of the model is that it expresses most of the term-structure-related uncertainty regarding bond returns as a vector product of $q(w)$ and a duration vector over a finite interval. By selecting portfolios with specific duration vectors, the effects of $q(w)$ may be controlled.

2.7 The need of Dynamic Global Portfolio Immunization Theorem

Many studies by Fisher and Weil [7], Bierwag [10], [11], Bierwag and Kaufman [7], and Khang [1] demonstrate that it is possible to immunize a portfolio of default-free assets against unexpected interest rate changes so that, at the end of the planning period, the investor will have at least the returns expected at purchase. However, this immunization strategy is applicable for the case in which the unexpected change of interest rate,assumed to be small, occurs one time only immediately after the purchase of the asset (bonds).

Obviously, these cases are not likely to resemble reality in at least three aspects:

1. The interest rate change is likely to occur at any time.

2. The interest rate change is likely to occur many times during the investor's planning period.

3. The interest rate change might be considerable.

Beiwarg [5] deals with the case in which the interest rate change may occur many times during the planning period, with each change occuring instantly after a time period begins anew, and shows that, provided the interest rate changes are small, an investor can , at the end of the planning period, realize the returns expected at purchase by constructing a portfolio whose duration and present value at the beginning of each period are equal to that of the obligations. Thus, Bierwag's [5] immunization theorem is a local theorem. It is found that the theorem cannot resemble reality since the paper only considers one kind of interest rate. But since liability interest rates are always different to the spot rates, this is not an accurate reflection of the reality. In the next chapter, the optimal immunization strategy in multi-period portfolio selection will be discussed.

Chapter 3

Multi-Period Portfolio Selection

3.1 Objective

Our objective is to find an applicable method which has fewer limitations than immunization and the other existing methods. Our method will overcome most of the limitations suffered by the immunization process.

As mentioned in previous chapters, the limitations of existing methods, such as the immunization method or the duration targeting method, can be summarized as follow:

1. The methods consider a one-period formulation.

2. The interest rate change occurs only one time.

3. All the methods adapt a one-factor model.

4. All the methods assume all interest rates to be the same.

However, the above assumptions are not true in the reality since:

1. Many investors always consider investing their money in a long term sense, which does not apply to one-period model.
2. It is well known that interest rate changes might occur multiple times, even within a short term period.
3. In reality, interest rates may not have a parallel shift. Thus, the interest rate curve may twist.
4. The liability rate, a bond's yield and spot rates are often different.

To overcome the limitations existing in the current literature, the interest rate should be allowed to change at different time intervals within the time period $[0, 1, 2, ..., T-1, T]$. As interest rates may change frequently within the whole time period $[0, 1, ..., T-1, T]$, the overall change of the interest rate might be large. Moreover, two different kinds of rate changes are considered: the change of the yield of bonds and the change of the liability rate. Moreover, instead of considering the shift of the spot rate curve, we consider the change of the yield of bonds and the change of liability rate to be probabilistic. In other words, we allow non-parallel shifts of interest rate curve in our model.

3.2 Dynamic Programming Formulation

The following is the main idea of our method:

At the start of each time interval, we are going to construct a portfolio out of two bonds with different durations. This portfolio will maximize the utility of increased wealth (wealth is defined to be the difference between the assets and liability) from the current time instance to the wealth at time T. As the portfolio will be reconstructed at the start of each time interval, it is a sign that dynamic programming has to be applied. As we mentioned before, we want to maximize the utility of increased wealth within a time period. The objective function we consider is:

$$V_0(W(0)) = Max\{E[U(W(T)) - U(W(0))]\} \tag{3.1}$$

As we want to express the change of wealth in time periods, equation (3.1) can be re-written as follows where the second-order taylor's expansion is used.

$$\begin{aligned} V_0(W(0)) &= Max\{E[U(W(T)) - U(W(0))]\} \\ &= Max\{\sum_{t=0}^{T-1} E[U(W(t+1)) - U(W(t))]\} \\ &\approx Max\{E[\sum_{t=0}^{T-1}[U'(W(t))\Delta W(t) + \frac{1}{2}U''(W(t))[\Delta W(t)]^2]]\} \\ &= Max\{\sum_{t=0}^{T-1} E\left[U'(W(t))\Delta W(t)\right] + \frac{1}{2}E\left[U''(W(t))[\Delta W(t)]^2\right]\} \end{aligned} \tag{3.2}$$

To make it clearer, we have the following conventions:

$A(t)$ is the assets value at time t.

$L(t)$ is the liability at time t.

$W(t)$ is the amount of wealth at the beginning of time interval $[t, t+1]$.

$\pi^{B1}(t)$ is the amount of bond 1 invested at time t.

$\pi^{B2}(t)$ is the amount of bond 2 invested at time t.

$B^k(t)$ is the price of bond k at the beginning of $[t, t+1]$, $k = 1, 2$.

$C^{B1}(t)$ is the coupon offered by bond 1 at time t.

$C^{B2}(t)$ is the coupon offered by bond 2 at time t.

$D^{B1}(t)$ is the modified duration of bond 1 at time t.

$D^{B2}(t)$ is the modified duration of bond 2 at time t.

$i^{A1}(t)$ is the yield of bond 1 at time t.

$i^{A2}(t)$ is the yield of bond 2 at time t.

$i^{L}(t)$ is the yield of the liability at time t.

$\Delta i^{A1}(t)$ is the change of yield of bond 1 at time t.

$\Delta i^{A2}(t)$ is the change of yield of bond 2 at time t.

$\Delta i^{L}(t)$ is the change of the liability rate.

Note that $\Delta i^{A1}(t)$ and $\Delta i^{A2}(t)$ have the same distribution with the same mean and standard deviation. For the sake of brevity, we simply use $\Delta i^{A}(t)$ in the future.

The coupon values $C^{B1}(t)$ and $C^{B2}(t)$ are fixed for all t. However, bond prices $B^1(t)$, $B^2(t)$ and the bond modified durations, $D^{B1}(t)$, $D^{B2}(t)$ are random. They depend on the realization of $i^{A1}(t)$ and $i^{A2}(t)$. When $i^{A1}(t)$ and $i^{A2}(t)$ are known, $B^1(t)$, $B^2(t)$, $D^{B1}(t)$, $D^{B2}(t)$ are uniquely determined, we have,

$$
\begin{aligned}
B^1(t) &= \sum_{k=1}^{T} \frac{C^{B1}(k)}{(1+i^{A1}(t))^k}. \\
B^2(t) &= \sum_{k=1}^{T} \frac{C^{B2}(k)}{(1+i^{A2}(t))^k}. \\
D^{B1}(t) &= \frac{1}{B^1(t)} \sum_{k=1}^{T} \frac{C^{B1}(k) \times k}{(1+i^{A1}(t))^k}. \\
D^{B2}(t) &= \frac{1}{B^2(t)} \sum_{k=1}^{T} \frac{C^{B2}(k) \times k}{(1+i^{A2}(t))^k}.
\end{aligned}
$$

Similarly, $L(t)$ is random before time t. Its value is uniquely determined by $i^L(t)$,

$$L(t) = L(0)(1 + i^L(t))^t$$

Suppose that an interest rate change occurs at time interval $[t, t+\epsilon]$, which changes the value of one particular bond. We now have the following,

$$\begin{aligned}
\Delta A(t) &= \pi^{B1}(t)\Delta B^1(t) + \pi^{B2}(t)\Delta B^2(t) \qquad (3.3)\\
\Delta W(t) &= \Delta A(t) - \Delta L(t)\\
&= \pi^{B1}(t)\Delta B^1(t) + \pi^{B2}(t)\Delta B^2(t) - \Delta i^L(t)L(t)\\
&\approx \pi^{B1}(t)(-D^{B1}(t)\Delta i^A(t)B^1(t))\\
&\quad + \pi^{B2}(t)(-D^{B2}(t)\Delta i^A(t)B^2(t)) - \Delta i^L(t)L(t) \qquad (3.4)
\end{aligned}$$

Going back to the objective function, we have to calculate $E[\Delta W(t)]$ and $E[\Delta W(t)^2]$.

First, we calculate $E[\Delta W(t)]$,

$$\begin{aligned}
E(\Delta W(t)) &= -\pi^{B1}(t)D^{B1}(t)\mu^A(t)B^1(t) - \pi^{B2}(t)D^{B2}(t)\mu^A(t)B^2(t)\\
&\quad -\mu^L(t)L(t) \qquad (3.5)
\end{aligned}$$

where $E[\Delta i^A(t)] = \mu^A(t)$ and $E[\Delta i^L(t)] = \mu^L(t)$.

Note that $\Delta i^A(t)$ and $\Delta i^L(t)$ are random variables following normal distribution with mean $\mu^A(t)$, $\mu^L(t)$ and standard deviation $\sigma^A(t)$, $\sigma^L(t)$ respectively.

As,

$$A(t) = \pi^{B1}(t)B^1(t) + \pi^{B2}(t)B^2(t)$$

we have,

$$\begin{aligned}
\pi^{B2}(t) &= \left(\frac{A(t) - \pi^{B1}(t)B^1(t)}{B^2(t)}\right) \\
&= \left(\frac{A(t)}{B^2(t)} - \frac{B^1(t)}{B^2(t)}\pi^{B1}(t)\right) \\
&= S(t) - U(t)\pi^{B1}(t) \qquad (3.6)
\end{aligned}$$

where

$$S(t) = \frac{A(t)}{B^2(t)} \qquad (3.7)$$

$$U(t) = \frac{B^1(t)}{B^2(t)} \qquad (3.8)$$

Substituting equation (3.6) into equation (3.5) yields,

$$
\begin{aligned}
E(\Delta W(t)) &= -\pi^{B1}(t)D^{B1}(t)\mu^{A}(t)B^{1}(t) - \pi^{B2}(t)D^{B2}(t)\mu^{A}(t)B^{2}(t) \\
&\quad -\mu^{L}(t)L(t) \\
&= -\pi^{B1}(t)D^{B1}(t)\mu^{A}(t)B^{1}(t) - \left(\frac{A(t) - \pi^{B1}(t)B^{1}(t)}{B^{2}(t)}\right) D^{B2}(t)\mu^{A}(t)B^{2}(t) \\
&\quad -\mu^{L}(t)L(t) \\
&= -\pi^{B1}(t)D^{B1}(t)\mu^{A}(t)B^{1}(t) - \left(A(t) - B^{1}(t)\pi^{B1}(t)\right) D^{B2}(t)\mu^{A}(t) \\
&\quad -\mu^{L}(t)L(t) \\
&= [-\pi^{B1}(t)D^{B1}(t)B^{1}(t) + B^{1}(t)\pi^{B1}(t)D^{B2}(t) - A(t)D^{B2}(t)]\mu^{A}(t) - L(t)\mu^{L}(t) \\
&= \{\pi^{B1}(t)B^{1}(t)[D^{B2}(t) - D^{B1}(t)] - A(t)D^{B2}(t)\}\mu^{A}(t) - L(t)\mu^{L}(t) \qquad (3.9)
\end{aligned}
$$

We now calculate $E[\Delta W(t)^2]$,

$$
\begin{aligned}
\Delta W(t) &= \pi^{B1}(t)(-D^{B1}(t)\Delta i^{A}(t)B^{1}(t)) + \pi^{B2}(t)(-D^{B2}(t)\Delta i^{A}(t)B^{2}(t)) \\
&\quad -\Delta i^{L}(t)L(t) \\
&= [\pi^{B1}(t)B^{1}(t)[D^{B2}(t) - D^{B1}(t)] - A(t)D^{B2}(t)][\Delta i^{A}(t)] \\
&\quad -L(t)[\Delta i^{L}(t)]
\end{aligned}
$$

$$
\begin{aligned}
\Delta W(t)^2 &= [\pi^{B1}(t)B^{1}(t)[D^{B2}(t) - D^{B1}(t)] - A(t)D^{B2}(t)]^2[\Delta i^{A}(t)]^2 \\
&\quad -2[L(t)][\pi^{B1}(t)B^{1}(t)[D^{B2}(t) - D^{B1}(t)] - A(t)D^{B2}(t)] \\
&\quad [\Delta i^{A}(t)][\Delta i^{L}(t)] + [L(t)]^2[\Delta i^{L}(t)]^2
\end{aligned}
$$

$$
\begin{aligned}
E[\Delta W(t)^2] &= [\pi^{B1}(t)B^{1}(t)[D^{B2}(t) - D^{B1}(t)] - A(t)D^{B2}(t)]^2 E\{[\Delta i^{A}(t)]\}^2 \\
&\quad -2[L(t)][\pi^{B1}(t)B^{1}(t)[D^{B2}(t) - D^{B1}(t)] - A(t)D^{B2}(t)] \\
&\quad E\{[\Delta i^{A}(t)][\Delta i^{L}(t)]\} + [L(t)]^2 E\{[\Delta i^{L}(t)]^2\}
\end{aligned}
\tag{3.10}
$$

Consider the following,

$$\begin{aligned}
E[(\Delta i^A(t))^2] &= E[\Delta i^A(t) - E(\Delta i^A(t))]^2 + 2\Delta i^A(t)E(\Delta i^A(t)) \\
&\quad -[E(\Delta i^A(t))]^2 \\
&= Var(\Delta i^A(t)) + (\mu^A(t))^2 \qquad (3.11)
\end{aligned}$$

$$E[(\Delta i^L(t))^2] = Var(\Delta i^L(t)) + (\mu^L(t))^2 \qquad (3.12)$$

$$\begin{aligned}
E(\Delta i^L(t)\Delta i^A(t)) &= E\{[\Delta i^L(t) - E(\Delta i^L(t))][\Delta i^A(t) - E(\Delta i^A(t))] \\
&\quad +\Delta i^L(t)E(\Delta i^A(t)) + \Delta i^A(t)E(\Delta i^L(t) \\
&\quad -E(\Delta i^A(t)E(\Delta^L(t))))\} \\
&= Cov(\Delta i^L(t), \Delta i^A(t)) + E(\Delta i^L(t))E(\Delta i^A(t)) \\
&= Cov(\Delta i^A(t), \Delta i^L(t)) + \mu^A(t)\mu^L(t) \qquad (3.13)
\end{aligned}$$

$Var(\Delta i^A(t))$ and $Var(\Delta i^L(t))$ are the variance of the change of the bond's yield and the variance of the change of the liability rate respectively. $Cov(\Delta i^A(t), \Delta i^L(t))$ is the covariance of the change of the bond's yield and the change of the liability rate.

We have,

$$
\begin{aligned}
E\{[\Delta W(t)]^2\} &= [\pi^{B1}(t)B^1(t)[D^{B2}(t) - D^{B1}(t)] - A(t)D^{B2}(t)]^2 \\
&\quad [(\mu^A(t))^2 + Var(\Delta i^A(t))] \\
&\quad -2L(t)[\pi^{B1}(t)B^1(t)[D^{B2}(t) - D^{B1}(t)] - A(t)D^{B2}(t)] \\
&\quad [Cov(\Delta i^A(t)\Delta i^L(t)) + \mu^A(t)\mu^L(t)] \\
&\quad +(L(t))^2[(Var(\Delta i^L(t)) + (\mu^L(t))^2] \qquad (3.14)
\end{aligned}
$$

In order to achieve an optimal policy, we will use dynamic programming to carry out backwards optimization. We have to find the optimal policy at time 0. In order to get dynamic programing formulation for this problem, we need to identify the corresponding **stage** variables, the **state** variable and the **decision** variables and the system transition equation:

Stage Variable: The stage variable in this problem here is simply the time instance where the investor can adjust his/her portfolio, t = Time t $(t = 1, 2, 3, \ldots, T-1, T)$.

Decision Variable: The decision variables at stage t are the amounts to be invested into bond 1 and bond 2, $\pi^{B1}(t)$ and $\pi^{B2}(t)$.

State Variable: When an investor makes his/her decision at stage t, he/she needs to know the current information about $A(t)$, $B^1(t)$, $B^2(t)$, $L(t)$, $D^{B1}(t)$, $D^{B2}(t)$.

Note that the bond prices $B^1(t)$, $B^2(t)$ and the bond durations $D^{B1}(t)$

and $D^{B2}(t)$ are uniquely determined when interest rates $i^{A1}(t)$ and $i^{A2}(t)$ are known; $L(t)$ is uniquely determined when $i^{L}(t)$ is known; and the determination of $A(t)$ requires information about decision taken at the previous stage, in addition to $B^1(t)$ and $B^2(t)$. Remember that the state is the minimum information set sufficient to make decisions. Thus, $A(t)$, $i^{A1}(t)$, $i^{A2}(t)$ and $i^{L}(t)$, the total asset, bond's yield and the liability rate at time t respectively, are qualified to serve as state variables.
The following are the state equations:

$$\begin{aligned}
i^{A1}(t+1) &= i^{A1}(t) + \Delta i^{A1}(t) \\
i^{A2}(t+1) &= i^{A2}(t) + \Delta i^{A2}(t) \\
i^{L}(t+1) &= i^{L}(t) + \Delta i^{L}(t) \\
A(t+1) &= \pi^{B1}(t)[B^1(t+1) + CB^1(t+1)] \\
&\quad + \pi^{B2}(t)[B^2(t+1) + CB^2(t+1)]
\end{aligned} \tag{3.15}$$

Where $B^1(t+1)$and $B^2(t+1)$ are determined by $i^{A}(t+1)$ and $i^{A2}(t+1)$.

Let V_t^* be the **wealth-to-go function** in dynamic programming, the following recursive relation holds:

$$
\begin{aligned}
& V_t^*(A(t), i^{A1}(t), i^{A2}(t), i^L(t)) \\
= & Max\sum_{k=t}^{T-1}[U'(W(k))E(\Delta W(t)) + \frac{1}{2}U''(W(k))E(\Delta W(t))^2] \\
= & MaxE\{U'(W(t))\Delta W(t) + \frac{1}{2}U''(W(t))(\Delta W(t))^2 \\
& +V_{t+1}^*(A(t+1), i^{A1}(t+1), i^{A2}(t+1), i^L(t+1))\} \qquad (3.16)
\end{aligned}
$$

3.3 Specific Situation

We should consider a strictly increasing concave utility function in order to ensure the existence of the optimal solution for the maximization problem.

Let us consider the following function as our utility function.

$$
\begin{aligned}
U(x) &= ln(x+k) \quad for \ \ x > -k+1 \\
&= 0 \qquad\qquad for \ x \leq -k+1 \qquad (3.17)
\end{aligned}
$$

Its first and the second derivative are as follow,

For $x > -k + 1$

$$\begin{aligned} U'(x) &= \frac{1}{x+k} \\ U''(x) &= -\frac{1}{(x+k)^2} \end{aligned}$$

For $x = -k + 1$

$$\begin{gathered} lim_{x \to (-k+1)_+} U'(x) = 1 \\ lim_{x \to (-k+1)_-} U'(x) = 1 \end{gathered} \tag{3.18}$$

For $x < -k + 1$

$$\begin{aligned} U'(x) &= 0 \\ U''(x) &= 0 \end{aligned}$$

Consider again the **objective function**,

$$V_0 = Max\{\sum_{t=0}^{T-1}[U'(W(t))E(\Delta W(t)) + \frac{1}{2}U''(W(t))E(\Delta W(t))^2)]\}$$

We perform the dynamic programming backward starting from stage T.

Stage T:

As we are maximizing the expected gain of utility of wealth from the current stage to the final stage, the current stage now being the final - the gain should be zero.

$$V_T = 0$$

Stage $T-1$:

This is a problem consisting of just the last two stages.

We have the **wealth-to-go function** as follow:

For $W(T-1) > -k+1$,

$$\begin{aligned}
V_{T-1} &= Max[V(T-1)] \\
&= MaxU'(W(T-1))E(\Delta W(T-1)) + \frac{1}{2}U''(W(T-1))E(\Delta W(T-1))^2 \\
&= Max[\frac{E(\Delta W(T-1))}{W(T-1)+k} - \frac{E(\Delta W(T-1))^2)}{[W(T-1)+k]^2}]
\end{aligned} \tag{3.19}$$

where,

$$\begin{aligned}
U'(W(T-1)) &= \frac{1}{W(T-1)+k} \\
U''(W(T-1)) &= -\frac{1}{[W(T-1)+k]^2}
\end{aligned}$$

and,

$$V(T-1) = \frac{E(\Delta W(T-1))}{W(T-1)+k} - \frac{E(\Delta W(T-1))^2}{[W(T-1)+k]^2} \tag{3.20}$$

For $W(T-1) \leq -k+1$

$$V_{T-1} = Max[V(T-1)] = 0$$

As in stage $T-1$, we have to find the optimal investment amount of bond 1 and bond 2 respectively. For given state variables $A(T-1)$, $i^{A1}(T-1)$, $i^{A2}(T-1)$ and $i^L(T-1)$, we maximize $V(T-1)$ with respect to $\pi^{B1}(T-1)$ and $\pi^{B2}(T-1)$.

We derive the optimal decision as follows.

$$
\begin{aligned}
A(T-1) &= \pi^{B1}(T-2)B^1(T-1) + \pi^{B2}(T-2)B^2(T-1) + \pi^{B1}(T-2)C^{B1}(T-1) \\
&\quad +\pi^{B2}(T-2)C^{B2}(T-1) \\
W(T-1) &= A(T-1) - L(T-1) \\
\Delta A(T-1) &= \pi^{B1}(T-1)(\Delta B^1(T-1)) + \pi^{B2}(T-1)(\Delta B^2(T-1)) \\
\Delta L(T-1) &= L(T-1)\Delta i^L(T-1) \\
\Delta W(T-1) &= \Delta A(T-1) - \Delta L(T-1) \\
&= \pi^{B1}(T-1)\left[\Delta B^1(T-1)\right] + \pi^{B2}(T-1)\left[\Delta B^2(T-1)\right] \\
&\quad -L(T-1)\left[\Delta i^L(T-1)\right] \\
&= \pi^{B1}(T-1)\left[-D^{B1}(T-1)\Delta i^A(T-1)B^1(T-1)\right] \\
&\quad +\pi^{B2}(T-1)[-D^{B2}(T-1)\Delta i^A(T-1)B^2(T-1)] \\
&\quad -L(T-1)\Delta i^L(T-1)
\end{aligned}
$$

Note that, as in (3.6)

$$
\pi^{B2}(T-1) = \ = \ S(T-1) - U(T-1)\pi^{B1}(T-1)
$$

Make use of the above equation, which has been mentioned before, we get the following,

$$
\begin{aligned}
E[\Delta W(T-1)] &= \{\pi^{B1}(T-1)B^1(T-1)[D^{B2}(T-1)-D^{B1}(T-1)] \\
&\quad -A(T-1)D^{B2}(T-1)\}\mu^A(T-1)-L(T-1)[\mu^L(T-1)]
\end{aligned} \tag{3.21}
$$

$$
\begin{aligned}
E[\Delta W(T-1)]^2 &= [\pi^{B1}(T-1)B^1(T-1)[D^{B2}(T-1)-D^{B1}(T-1)] \\
&\quad -A(T-1)D^{B2}(T-1)]^2[(\mu^A(T-1))^2+Var(\Delta i^A(T-1))] \\
&\quad -2L(T-1)[\pi^{B1}(T-1)B^1(T-1)[D^{B2}(T-1) \\
&\quad -D^{B1}(T-1)]-A(T-1)D^{B2}(T-1)] \\
&\quad [Cov(\Delta i^A(T-1)\Delta i^L(T-1))+\mu^A(T-1)\mu^L(T-1)] \\
&\quad +(L(T-1))^2[Var(\Delta i^L(T-1))+(\mu^L(T-1))^2]
\end{aligned} \tag{3.22}
$$

For $W(T-1) > -k+1$, we differentiate the equation (3.19) with respect to $\pi^{B1}(T-1)$ and set it equal to zero, we have,

$$
\begin{aligned}
[V(T-1)]' &= \frac{dV(T-1)}{d\pi^{B1(T-1)}} \\
&= \frac{-D^{B1}(T-1)B^1(T-1)\mu^A(T-1) + B^1(T-1)D^{B2}(T-1)\mu^A(T-1)}{W(T-1)+k} \\
&\quad -\frac{[E(\Delta W(T-1)^2)]'}{[W(T-1)+k]^2}
\end{aligned} \tag{3.23}
$$

where $[E(\Delta W(T-1)^2)]'$ denotes

$$
\frac{dE(\Delta W(T-1)^2)}{d\pi^{B1}(T-1)}
$$

and it is equal to

$$
\begin{aligned}
[E(\Delta W(T-1)^2)]' &= 2[Var(\Delta i^A(T-1)) + (\mu^A(T-1))^2] \\
&\quad [\pi^{B1}(T-1)B^1(T-1)[D^{B2}(T-1) - D^{B1}(T-1)] \\
&\quad -A(T-1)D^{B2}(T-1)][B^1(T-1)[D^{B2}(T-1) - D^{B1}(T-1)]] \\
&\quad -2L(T-1)[Cov(\Delta i^A(T-1), \Delta i^L(T-1)) + \mu^A(T-1)\mu^L(T-1)] \\
&\quad [D^{B2}(T-1) - D^{B1}(T-1)]B^1(T-1) \\
&= 2B^1(T-1)[D^{B2}(T-1) - D^{B1}(T-1)]\{[Var(\Delta i^A(T-1)) \\
&\quad +(\mu^A(T-1))^2][\pi^{B1}(T-1)B^1(T-1)[D^{B2}(T-1) - D^{B1}(T-1)] \\
&\quad -A(T-1)D^{B2}(T-1)] - L(T-1)[Cov(\Delta i^A(T-1), \Delta i^L(T-1)) \\
&\quad +\mu^A(T-1)\mu^L(T-1)]\}
\end{aligned} \tag{3.24}
$$

therefore, for $W(T-1) > -k+1$,

$$
\begin{aligned}
[V(T-1)]' &= \frac{dV(T-1)}{d\pi^{B1(T-1)}} \\
&= \frac{-D^{B1}(T-1)B^1(T-1)\mu^A(T-1) + B^1(T-1)D^{B2}(T-1)\mu^A(T-1)}{W(T-1)+k} \\
&\quad - \frac{[E(\Delta W(T-1)^2)]'}{[W(T-1)+k]^2} \\
&= \frac{B^1(T-1)\mu^A(T-1)(D^{B2}(T-1) - D^{B1}(T-1))}{W(T-1)+k} \\
&\quad - \frac{1}{[W(T-1)+k]^2} B^1(T-1)[D^{B2}(T-1) - D^{B1}(T-1)] \\
&\quad \{[Var(\Delta i^A(T-1)) + (\mu^A(T-1))^2][\pi^{B1}(T-1)B^1(T-1) \\
&\quad [D^{B2}(T-1) - D^{B1}(T-1)] - A(T-1)D^{B2}(T-1)] - L(T-1) \\
&\quad [Cov(\Delta i^A(T-1), \Delta i^L(T-1)) + \mu^A(T-1)\mu^L(T-1)]\}
\end{aligned}
\tag{3.25}
$$

At stage $T-1$, the optimal number of shares of bond 1 which should be held is

$$
\begin{aligned}
\pi^{B1}(T-1)^* &= \frac{1}{B^1(T-1)[D^{B2}(T-1)-D^{B1}(T-1)]} \\
&\quad \{\frac{\mu^A(T-1)[W(T-1)+k]}{[Var(\Delta i^A(T-1))+(\mu^A(T-1))^2]} \\
&\quad + \frac{L(T-1)[Cov(\Delta i^A(T-1),\Delta i^L(T-1))+\mu^A(T-1)\mu^L(T-1)]}{[Var(\Delta i^A(T-1))+(\mu^A(T-1))^2]} \\
&\quad + A(T-1)D^{B2}(T-1)\} \qquad (3.26)
\end{aligned}
$$

On the other hand, the optimal number of shares of *bond* 2 should be equal to:

$$\pi^{B2}(T-1)^* = S(T-1) - U(T-1)\pi^{B1}(T-1)^* \qquad (3.27)$$

For $W(T-1) \leq -k+1$, we have

$$
\begin{aligned}
\pi^{B1}(T-1)^* &= 0 \\
\pi^{B2}(T-1)^* &= 0
\end{aligned}
$$

We denote V^*_{T-1} to be the optimal cost-to-go function at stage $T-1$. By substituting the optimal solution obtained above into (3.20), we can obtain V^*_{T-1} at stage $T-1$.

Stage $T-2$:

The **wealth-to-go function** at stage $T-2$ is the sum of the current wealth increase at stage $T-2$ and the **expected** wealth-to-go at stage $T-1$, $For W(T-2) > -k+1$

$$
\begin{aligned}
V_{T-2} &= Max[V(T-2)] \\
&= Max[\frac{E(\Delta W(T-2))}{W(T-2)+k} - \frac{E(\Delta W(T-2))^2)}{[W(T-2)+k]^2} \\
&\quad +E(V^*_{T-1}(A(T-1), i^A(T-1), i^L(T-1)))]
\end{aligned}
$$

where,

$$
\begin{aligned}
V(T-2) &= \frac{E(\Delta W(T-2))}{W(T-2)+k} - \frac{E(\Delta W(T-2))^2)}{[W(T-2)+k]^2} \\
&\quad +E(V^*_{T-1}(A(T-1), i^A(T-1), i^L(T-1)))
\end{aligned}
\tag{3.28}
$$

For $W(T-2) \leq -k+1$

$$
\begin{aligned}
V_{T-2} &= Max[V(T-2)] \\
&= Max[E(V_{T-1}^*(A(T-1), i^A(T-1), i^L(T-1)))] \\
&= E(V_{T-1}^*(A(T-1), i^A(T-1), i^L(T-1))) \qquad (3.29)
\end{aligned}
$$

We have to decide the optimal amounts of bond one and bond two to be invested at stage $T-2$. The decision variables at stage $T-2$ should be $\pi^{B1}(T-2)$ and $\pi^{B2}(T-2)$.

State Variable at stage $T-2$:

$A(T-2)$, $i^{A1}(T-2)$, $i^{A2}(T-2)$ and $i^L(T-2)$ are the state variables at this stage. Go back to the objective fuction (3.28), we have to obtain $E(V_{T-1}^*)$ at stage $T-2$. Obtaining $E(V_{T-1}^*)$ is a difficult task, as the equation is not a linear equation of $\Delta i^A(T-2)$ and $\Delta i^L(T-2)$. Therefore, the usage of Taylor's expansion would help to solve the problem.

At stage $T-1$, we can obtain the optimal wealth-to-go function(shown in the previous stage), which is a function of $A(T-1)$,$L(T-1)$, $B^1(T-1)$, $B^2(T-1)$, $D^{B1}(T-1)$ and $D^{B2}(T-1)$, that is,

$$
\begin{aligned}
& V_{T-1}^*(A(T-1), i^{A1}(T-1), i^{A2}(T-1), i^L(T-1)) \\
= \ & f[A(T-1), L(T-1), B^1(T-1) \\
& , B^2(T-1), D^{B1}(T-1), D^{B2}(T-1)] \qquad (3.30)
\end{aligned}
$$

Note that at stage $T-2$, $A(T-1)$,$L(T-1)$, $B^1(T-1)$, $B^2(T-1)$, $D^{B1}(T-1)$

and $D^{B2}(T-1)$ are all random variables that depend on $i^{A1}(T-1)$, $i^{A2}(T-1)$ and $i^{L}(T-1)$ and also $\pi^{B1}(T-2)$ and $\pi^{B2}(T-2)$. As a result, the function (3.30) can actually be re-written at stage $T-2$ as follow:

$$\begin{aligned}
& V^*_{T-1} \\
= \ & f[A(T-1), L(T-1), B^1(T-1), B^2(T-1), D^{B1}(T-1), D^{B2}(T-1)] \\
= \ & g(i^{A1}(T-1), i^{A2}(T-1), i^{L}(T-1), \pi^{B1}(T-2), \pi^{B2}(T-2)) \qquad (3.31)
\end{aligned}$$

For the sake of brevity, we tackle the above function by applying the first order Taylor's expansion around $i^{A1}(T-2)$,$i^{A2}(T-2)$,$i^{L}(T-2)$:

$$\begin{aligned}
& g(i^{A1}(T-1), i^{A2}(T-1), i^{L}(T-1), \pi^{B1}(T-2), \pi^{B2}(T-2)) \\
\approx \ & g(i^{A1}(T-2), i^{A2}(T-2), i^{L}(T-2), \pi^{B1}(T-2), \pi^{B2}(T-2)) \\
& + \nabla g(i^{A1}(T-2), i^{A2}(T-2), i^{L}(T-2), \pi^{B1}(T-2), \pi^{B2}(T-2)) \\
& \bullet(\Delta i^{A}(T-2), \Delta i^{A}(T-2), \Delta i^{L}(T-2))^{T} \qquad (3.32)
\end{aligned}$$

As a result, the most difficult part of determining the expected value of $[V^*_{T-1}]$ has been changed to the much easier problem of finding the expected value of

$$g(i^{A1}(T-2), i^{A2}(T-2), i^{L}(T-2)) + \nabla g(i^{A1}(T-2), i^{A2}(T-2), i^{L}(T-2))$$
$$\bullet(\Delta i^{A}(T-2), \Delta i^{A}(T-2), \Delta i^{L}(T-2))^{T}$$

That is,

$$\begin{aligned}
& E[g(i^{A1}(T-2), i^{A2}(T-2), i^{L}(T-2), \pi^{B1}(T-2), \pi^{B2}(T-2)) \\
& + \nabla\, g(i^{A1}(T-2), i^{A2}(T-2), i^{L}(T-2), \pi^{B1}(T-2), \pi^{B2}(T-2)) \\
& \bullet(\Delta i^{A}(T-2), \Delta i^{A}(T-2), \Delta i^{L}(T-2))^{T}] \\
= \; & g(i^{A1}(T-2), i^{A2}(T-2), i^{L}(T-2), \pi^{B1}(T-2), \pi^{B2}(T-2)) \\
& + \nabla\, g(i^{A1}(T-2), i^{A2}(T-2), i^{L}(T-2), \pi^{B1}(T-2), \pi^{B2}(T-2)) \\
& \bullet(\mu^{A}(T-2), \mu^{A}(T-2), \mu^{L}(T-2))^{T} \qquad (3.33)
\end{aligned}$$

Finally, we are ready to find the optimal solution $\pi^{B1}(T-2)^{*}$ and $\pi^{B2}(T-2)^{*}$ by maximizing (3.28). The derivation process is similar to stage $T-1$.

Stage t: The **wealth-to-go funcion** is as follow:

$$\begin{aligned} V_t &= Max[V(t)] \\ &= Max[\frac{E(\Delta W(t))}{W(t)+k} + \frac{E(\Delta W(t))^2)}{[W(t)+k]^2} + E(V_{t+1}^*)] \end{aligned} \quad (3.34)$$

The decision variables are $\pi^{B1}(t)$ and $\pi^{B2}(t)$. The state variables are $A(t)$, $i^{A1}(t)$, $i^{A2}(t)$ and $i^L(t)$ which fluctuate at every stage.

Notes at stage t: $E(V_{t+1}^*)$ is determined by applying Taylor's expansion to V_{t+1}^* around $i^{A1}(t)$, $i^{A2}(t)$,$i^L(t)$. And by similar derivation as done at stage $T-2$, we get $\pi^{B1}(t)^*$ and $\pi^{B2}(t)^*$, which are the optimal number of shares to be held for bond one and bond two, respectively.

3.4 Summary of Implementation Results

As shown in appendix 1, a matlab program, designed to mirror a realistic situation of two-period portfolio selection by two different bonds, was constructed in order to immunize against the existing obligation at the start of the third stage.

Briefly, the situation is as follow:

The time horizon we are considering involves three stages, from stage 0 to stage 2. At stage 0, one would have \$120000000 together with an obligation of \$110000000 which is due at stage 2. There are two bonds we can invest. The information of the bonds and the liability is as follows:

Bond One offers three bond payments on stages 1, 2, and 3 while stage 3 is the maturity date of that bond. The payments of the bond are \$50, \$50 and \$100 at stage 1, 2 and 3, respectively. The yield to maturity of Bond One is 5% and the yield is allowed to change stochastically, following the normal distribution with mean 0.001 and standard deviation 0.001 at the beginning of each stage.

Bond Two offers three bond payments as well. They are \$10, \$10 and \$100 at stages 1, 2 and 3, respectively. The yield to maturity of Bond two is 6% and will change stochastically following the normal distribution with mean 0.001 and standard deviation 0.001 at the beginning of each stage.

The liability amount at stage 0 is \$110000000. The forward rate each year is assumed to be 11% from stage 0 to stage 1 and to change stochastically following the normal distribution with mean 0.01 and standard deviation 0.01 at the beginning of each stage.

The final results are plausible. Investing by the dynamic immunization strategy in this three-stage problem can return about $\$7 \times 10^{11}$ while the investing by the traditional immunization method can return a wealth amount of about $\$4 \times 10^{07}$.

We provide some computational results as follow, where WEALTH I denotes the wealth return by the Multiperiod Immunization Method and WEALTH II denotes the wealth return by the Duration Matching Method. Four tables are given for four different realizations of the interest rate changes.

Note the third and the fourth row of the table denotes the amount should be invest in Bond One and Bond two respectively at stage 0 and stage 2.

	STAGE 0	STAGE I	STAGE II	STAGE III
BOND I	\$0	\$50	\$50	\$100
BOND II	\$0	\$10	\$10	\$100
BOND I HOLDINGS	$-\$1.326 \times 10^{06}$	$\$1.2305 \times 10^{06}$		
BOND II HOLDINGS	$\$3.4001 \times 10^{06}$	$-\$9.6942 \times 10^{5}$		
WEALTH I	$\$10^{07}$	$-\$2.2037 \times 10^{07}$	$\$2.8543 \times 10^{11}$	
WEALTH II	$\$10^{07}$		$\$3.8661 \times 10^{07}$	

Table 3.1: Simulation One

	STAGE 0	STAGE I	STAGE II	STAGE III
BOND I	\$0	\$50	\$50	\$100
BOND II	\$0	\$0	\$10	\$100
BOND I HOLDINGS	$-\$3.5127 \times 10^{05}$	$\$1.0906 \times 10^{06}$		
BOND II HOLDINGS	$\$1.8630 \times 10^{06}$	$-\$7.9372 \times 10^{5}$		
WEALTH I	$\$10^{07}$	$-\$1.6471 \times 10^{07}$	$\$1.5671 \times 10^{11}$	
WEALTH II	$\$10^{07}$		$\$5.1749 \times 10^{07}$	

Table 3.2: Simulation Two

	STAGE 0	STAGE I	STAGE II	STAGE III
BOND I	\$0	\$50	\$50	\$100
BOND II	\$0	\$0	\$0	\$100
BOND I HOLDINGS	\$0	$\$1.0415 \times 10^{06}$		
BOND II HOLDINGS	$\$1.3101 \times 10^{06}$	$-\$7.6531 \times 10^{5}$		
WEALTH I	$\$10^{07}$	$-\$1.9796 \times 10^{08}$	$\$1.9214 \times 10^{11}$	
WEALTH II	$\$10^{07}$		$\$7.417 \times 10^{07}$	

Table 3.3: Simulation Three

	STAGE 0	STAGE I	STAGE II	STAGE III
BOND I	\$0	\$80	\$80	\$100
BOND II	\$0	\$0	\$0	\$100
BOND I HOLDINGS	\$0	$\$6.5076 \times 10^{05}$		
BOND II HOLDINGS	$\$1.3101 \times 10^{06}$	$-\$3.6622 \times 10^{5}$		
WEALTH I	$\$10^{07}$	$-\$1.2206 \times 10^{08}$	$\$9.076 \times 10^{11}$	
WEALTH II	$\$10^{07}$		$\$7.4 \times 10^{07}$	

Table 3.4: Simulation Four

Chapter 4

Summary

This thesis introduces a new method of immunization based upon multiple-stage, not single-stage immunization. Previous immunization methods, such as duration-matching and duration-targeting, are single stage immunization methods which require, as basic tenets of their theses, small interest rate change occurring only once. They also assume that the interest rate change operates under a one-factor model. This is, without question, an unreasonable description of reality since interest rate changes periodically. As well, the interest rate may not change so quietly; in fact, it could be very well rise or fall considerably. Nor does the assumption that such interest rate changes adhere to a parallel shift in any way resemble reality. Most of the extant literature also assumes that all interest rates are the same, which is again, an erroneous assumption since the liability rate and the spot rate may be different.

The method discussed in this thesis addresses the problems mentioned in above. We assume that the interest rate might change periodically - and considerably as well. Non-parallel interest rate changes are also allowed in our model. Finally, our model takes into consideration three kinds of interest rates - Bond One's yield, Bond Two's yield and the liability rate - making this new method a more accurate description of reality.

Though the dynamic multi-period portfolio selection method seems plausible, there are still some weaknesses - which could be improved - existing in the model.

Peter Ove [3] has proven that convexity is an important tool for controlling the price risk caused by shifts in interest rates. Therefore, it is worthwhile to extend our work by including a consideration of the convexity while approximating the yield change of the bonds.

The formulation is shown below with the conventions the same as mentioned in the previous chapter.

Let

$\zeta^{B1}(t)$ be the convexity of bond 1 at time t.

$\zeta^{B2}(t)$ be the convexity of bond 2 at time t.

Then the changes in the asset and the wealth can be expressed as follows,

$$\begin{aligned}
\Delta A(t) &= \pi^{B1}(t)\Delta B^1(t) + \pi^{B2}(t)\Delta B^2(t) \\
\Delta W(t) &= \Delta A(t) - \Delta L(t) \\
&= \pi^{B1}(t)\Delta B^1(t) + \pi^{B2}(t)\Delta B^2(t) - \Delta L(t) \\
&= \pi^{B1}(t)\Delta B^1(t) + \pi^{B2}(t)\Delta B^2(t) - \Delta i^L(t)L(t) \\
&\approx \pi^{B1}(t)(-D^{B1}(t)\Delta i^{A1}(t)B^1(t) + \zeta^{B1}(t)(\delta i^{A1}(t)^2)B^1(t)) \\
&\quad +\pi^{B2}(t)(-D^{B2}(t)\Delta i^{A2}(t)B^2(t) + \zeta^{B2}(t)(\delta i^{A2}(t)^2)B^2(t)) \\
&\quad -\Delta i^L(t)L(t) \qquad (4.1)
\end{aligned}$$

With this modification, we believe that the generated portfolio will bring a more steady and higher return for investors.

There are no free lunches in the world, therefore, whatever portfolio selection method is used, a risk will always exist. For example, under the duration-matching method, a portfolio can no longer immunize against a liability if the interest rate does not change at all.

Under the dynamic portfolio selection method, the number of the bonds that could be sold short, as well the number of bonds that could be held, are unlimited. In other words, the number of Bond One and Bond Two available to the investors is unlimited. Therefore, they still suffer a default risk as the bond issuer may default for some reason. To insure vis-a-vis such risk, future works should limit the number of Bond One and Bond Two to a quantifiable amount.

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Appendix A

Matlab Program of the Dynamic Portfolio Selection

```
%==============================================================%
%===== FILE: stage1.m =========================================%
%==============================================================%
binfo;
clear;
load binfo;

syms piB11 piB21 piB12 piB22 yA1 yA2 yL

%==============================================================%
%----- sigmaA IS THE STANDARD DEVIATION -----------------------%
%----- OF THE CHANGE OF THE BOND'S YIELD ----------------------%
%==============================================================%
%----- sigmaL IS THE STANDARD DEVIATION -----------------------%
%----- OF THE CHANGE OF THE LIABILITY RATE --------------------%
%==============================================================%
%----- miuA IS THE MEAN OF THE CHANGE OF THE BOND'S YIELD ---%
%==============================================================%
%----- miuL IS THE MEAN OF THE CHANGE OF THE LIABILITY RATE -%
%==============================================================%
%----- rowAL IS THE CORRELATION COEFFICEIENT BETWEEN --------%
%----- CHANGE OF BOND YIELD AND CHANGE OF LIABILITY RATE ----%
%==============================================================%

sigmaA=0.001;
sigmaL=0.001;
```

```
miuA=0.001;
miuL=0.001;
rowAL=1;
k=1000000

%==============================================================%
%----- yieldA1(i) IS BOND ONE'S YIELD AT STAGE "i" ----------%
%==============================================================%
%----- yieldA2(i) IS BOND TWO'S YIELD AT STAGE "i" ----------%
%==============================================================%
%----- AT HERE, WE ASSUME THAT BOND YIELDS ------------------%
%----- ARE CHANGED PARALLELY AT EACH STAGE ------------------%
%----- AND THE CHANGES ARE STOCHASTIC -----------------------%
%==============================================================%

yieldA1(1)=0.05;
yieldA2(1)=0.06;
yieldA1(2)=yieldA1(1)+2*(rand(1)-0.5)/1000;
yieldA2(2)=yieldA2(1)+2*(rand(1)-0.5)/1000;
yieldA1(3)=yieldA1(2)+2*(rand(1)-0.5)/1000;
yieldA2(3)=yieldA2(2)+2*(rand(1)-0.5)/1000;

%==============================================================%
%----- ratel(i) is the liability ---------------------------%
%----- forward rate from i to i+1 --------------------------%
%==============================================================%

ratel(1)=0.11;
ratel(2)=ratel(1)+2*(rand(1)-0.5)/100;

%==============================================================%
%----- B11 IS THE BOND PRICE OF BOND 1 AT STAGE 1 -----------%
%==============================================================%

B11=0;
for i=1:t1;
B11=CB1(i)/((1+yieldA1(1))^(i-1))+B11;
end;
B21;

%==============================================================%
```

```
%----- DB11 IS THE DURATION OF BOND 1 AT STAGE 1 ------------%
%============================================================%

DB11=0;
for i=1:t1;
DB11=CB1(i)*(i-1)/((1+yieldA1(1))^(i-1))+DB11;
end;
DB11=DB11/B11;

%============================================================%
%----- DB21 IS THE DURATION OF BOND 2 AT STAGE 1 ------------%
%============================================================%

DB21=0;
for i=1:t2;
DB21=CB2(i)*(i-1)/((1+yieldA2(1))^(i-1))+DB21;
end;
DB21=DB21/B21;

%============================================================%
%----- STAGE 2 STUFF AT STAGE 1 -----------------------------%
%============================================================%

%============================================================%
%----- B12 IS BOND PRICE OF BOND 1 AT STAGE 2 ---------------%
%============================================================%

B12=(((B11-CB1(1))*(1+yieldA1(1))-CB1(2))*(1+yieldA1(1))-CB1(3
))*(1+yieldA1(1))/(1+yA1)^2+CB1(3)/(1+yA1)+CB1(2);

%============================================================%
%----- B22 IS THE BOND PRICE OF BOND 2 AT STAGE 2 -----------%
%============================================================%

B22=(((B21-CB2(1))*(1+yieldA2(1))-CB2(2))*(1+yieldA2(1))-CB2(3
))*(1+yieldA2(1))/(1+yA2)^2+CB2(3)/(1+yA2)+CB2(2);

%============================================================%
%----- DB12 IS THE DURATION OF BOND 1 AT STAGE 2 ------------%
%============================================================%
```

```
DB12=(((DB11*B11-CB1(1)*1)*(1+yieldA1(1))-CB1(2)*2)*(1+yieldA1
(1))-CB1(3)*3)*(1+yieldA1(1))/(1+yA1)^2+CB1(3)*3/(1+yA1)+CB1(2
)*2;

%==============================================================%
%----- DB22 IS THE DURATION OF BOND 2 AT STAGE 2 ------------%
%==============================================================%

DB22=(((DB21*B21-CB2(1)*1)*(1+yieldA2(1))-CB2(2)*2)*(1+yieldA2
(1))-CB2(3)*3)*(1+yieldA2(1))/(1+yA2)^2+CB2(3)*3/(1+yA2)+CB2(2
)*2;

%==============================================================%
%----- piB21 IS THE NUMBER OF SHARES OF BOND 2 --------------%
%----- SHOULD BE HOLD AT STAGE 1 ----------------------------%
%==============================================================%
%----- piB11 IS THE NUMBER OF SHARES OF BOND 1 --------------%
%----- SHOULD BE HOLD AT STAGE 1 ----------------------------%
%==============================================================%
%----- CA DENOTES THE CAPTITAL AT STAGE 1 -------------------%
%==============================================================%
%----- CL DENOTES THE CAPTITAL AT STAGE 1 -------------------%
%==============================================================%
%----- CB1(i) DENOTES THE COUPON OFFERED --------------------%
%----- BY BOND 1 AT STAGE "i" -------------------------------%
%==============================================================%
%----- CB2(i) DENOTES THE COUPON OFFERED --------------------%
%----- BY BOND 2 AT STAGE "i" -------------------------------%
%==============================================================%
%----- A2 IS THE ASSETS VALUE AT STAGE 2 --------------------%
%==============================================================%
%----- L2 IS THE LIABILITY AT STAGE 2 -----------------------%
%==============================================================%
%----- W2 IS THE WEALTH AT STAGE 2 --------------------------%
%==============================================================%
%----- yL IS THE ONE YEAR LIABILITY RATE --------------------%
%==============================================================%

piB21=(CA-piB11*B11)/B21;
A2=piB11*B12+piB21*B22+piB11*CB1(2)+piB21*CB2(2);
L2=CL*(1+yL);
```

```
W2=A2-L2;

%==============================================================%
%----- piB12 IS THE NUMBER OF SHARES OF BOND 1 --------------%
%----- SHOULD BE HOLD AT STAGE 2 ----------------------------%
%==============================================================%
%----- piB22 IS THE NUMBER OF SHARES OF BOND 2 --------------%
%----- SHOULD BE HOLD AT STAGE 2 ----------------------------%
%==============================================================%
piB12=((W2+k)*miuA+L2*(rowAL*sigmaA*sigmaL+miuA*miuL))/(B12*(DB22-
DB12)*(sigmaA^2+miuA^2))+A2*DB22/(B12*(DB22-DB12))

piB22=(A2-piB12*B12)/B22;

%==============================================================%
%----- EdeltaW DENOTES THE EXPECTED VALUE -------------------%
%----- OF THE CHANGE OF W(WEALTH) ---------------------------%
%==============================================================%
%----- THE DERIVATION WAS SHOWN IN THE THESIS ---------------%
%==============================================================%
%----- EdeltaW2 DENOTES THE EXPECTED VALUE OF ---------------%
%==============================================================%
%----- THE CHANGE OF W^2(SQUARE OF WEALTH)-------------------%
%==============================================================%
%----- THE DERIVATION WAS SHOWN IN THE THESIS ---------------%
%==============================================================%

EdeltaW=piB12*(-DB12*miuA*B12)+piB22*(-DB22*miuA*B22)-L2*miuL;

EdeltaW2=(-piB12*DB12*B12+B12*piB12*DB22-A2*DB22)^2*(sigmaA^2+
miuA^2)-2*(L2)*(rowAL*sigmaA*sigmaL+miuA*miuL)*(-piB12*B12*DB1
2+B12*DB22-A2*DB22)+L2*(sigmaL^2+miuL^2);

%==============================================================%
%----- V2 IS THE COST-TO-GO FUNCTION AT STAGE 2 -------------%
%==============================================================%
V2=EdeltaW/(W2+k)-EdeltaW2^2/(2*(W2+k)^2);


%==============================================================%
%----- WE NEED THE FOLLOWING FOR THE TAYLOR'S EXPANSION -----%
```

```
%============================================================%

dA1=diff(V2,yA1);
dA1=subs(dA1,yA1,yieldA1(1));
dA1=subs(dA1,yA2,yieldA2(1));
dA1=subs(dA1,yL,ratel(1));

dA2=diff(V2,yA2);
dA2=subs(dA2,yA1,yieldA1(1));
dA2=subs(dA2,yA2,yieldA2(1));
dA2=subs(dA2,yL,ratel(1));

dL=diff(V2,yL);
dL=subs(dL,yA1,yieldA1(1));
dL=subs(dL,yA2,yieldA2(1));
dL=subs(dL,yL,ratel(1));

V2=EdeltaW/(W2+k)-EdeltaW2^2/(2*(W2+k)^2);
V2=subs(V2,yA1,yieldA1(1));
V2=subs(V2,yA2,yieldA2(1));
V2=subs(V2,yL,ratel(1));

%============================================================%
%----- EV2 DENOTES THE EXPECTED VALUE OF -------------------%
%----- THE STAGE 2 COST-TO-GO FUNCTION AT STAGE 1 ----------%
%============================================================%

EV2=V2+[dA1,dA2,dL]*[miuA;miuA;miuL];

%============================================================%
%----- W1 IS THE WEALTH AT STAGE 1 -------------------------%
%============================================================%

W1=CA-CL/(1+ratel(2))^2;

%============================================================%
%----- A1 IS THE ASSETS AMOUNT  AT STAGE 1, ----------------%
%----- WHICH IS EQUAL TO THE INITIAL CAPITAL ---------------%
%============================================================%
A1=CA;
```

```
%==============================================================%
%----- L1 IS THE LIABILITY AMOUNT AT STAGE 1, ---------------%
%----- WHICH IS EQUAL TO THE INITIAL LIABILITY --------------%
%==============================================================%

L1=CL;

EdeltaW=piB11*(-DB11*miuA*B11)+piB21*(-DB21*miuA*B21)-L1*miuL;
EdeltaW2=(-piB11*DB11*B11+B11*piB11*DB21-A1*DB21)^2*(sigmaA^2+
miuA^2)-2*(L1)*(rowAL*sigmaA*sigmaL+miuA*miuL)*(-piB11*B11*DB1
1+B11*DB21-A1*DB21)+L1*(sigmaL^2+miuL^2);


%==============================================================%
%----- V1 DENOTES THE COST-TO-GO FUNCTION AT STAGE 1, -------%
%----- WHICH IS READY FOR MAXIMIZATION NOW ------------------%
%==============================================================%

V1=EdeltaW/(W1+k)-EdeltaW2/(2*(W1+k)^2)+EV2;

%==============================================================%
%----- THIS IS THE MAXIMIZATION PROCESS ---------------------%
%==============================================================%

equation1=diff(V1,piB11);

piB11=numeric(solve(equation1,piB11));

%==============================================================%
%----- THIS IS THE OPTIMAL PORTION IN NUMERICAL FORM --------%
%==============================================================%

piB21=(A1-piB11*B11)/B21;

piB11=piB11(1)
piB21=piB21(1)

save stage1 piB11 piB21 yieldA1 yieldA2;
clear
stage2;
```

```
%==============================================================%
%===== FILE: stage2.m ==========================================%
%==============================================================%

load rinfo
load binfo
load stage1

%==============================================================%
%----- STAGE 2 -------------------------------------------------%
%==============================================================%
%----- PRICE OF BOND 1 AT STAGE 2 -----------------------------%
%==============================================================%

B12=0;
for i=2:t1;
B12=CB1(i)/((1+yieldA1(2))^(i-1))+B12;
end;
B12;

%==============================================================%
%----- PRICE OF BOND 2 AT STAGE 2 -----------------------------%
%==============================================================%

B22=0;
for i=2:t2;
B22=CB2(i)/((1+yieldA2(2))^(i-1))+B22;
end;
B22

%==============================================================%
%----- DURATION OF BOND 1 AT STAGE 2 --------------------------%
%==============================================================%

DB12=0;
for i=2:t1;
DB12=CB1(i)*(i-1)/((1+yieldA1(2))^(i-1))+DB12;
end;
DB12=DB12/B12;

%==============================================================%
```

```
%----- DURATION OF BOND 2 AT STAGE 2 ----------------------%
%===========================================================%

DB22=0;
for i=2:t2;
DB22=CB2(i)*(i-1)/((1+yieldA2(2))^(i-1))+DB22;
end;
DB22=DB22/B22;

%===========================================================%
%----- THESE VALUES HAD ALL BEEN INTRODUCED -----------------%
%----- AT THE PROGRAM "STAGE1" ------------------------------%
%===========================================================%

sigmaA=0.0001;
sigmaL=0.0001;
miuA=0.001;
miuL=0.001;
rowAL=1;

%===========================================================%
%----- A2 IS THE ASSETS AMOUT AT STAGE 2 -------------------%
%===========================================================%

A2=piB11*B12+piB11*CB1(2)+piB21*B12+piB21*CB2(2)

%===========================================================%
%----- L2 IS THE LIABILITY AMOUNT AT STAGE 2 ---------------%
%===========================================================%

L2=CL*(1+ratel(2));

%===========================================================%
%----- W2 IS THE WEALTH AMOUNT AT STAGE 2 ------------------%
%===========================================================%

W2=A2-L2;

%===========================================================%
%----- THESE TWO ARE THE OPTIMAL PORTION OF -----------------%
%----- BOND 1 AND BOND 2 AT STAGE 2 ------------------------%
```

```
%===============================================================%

piB12=(W2*miuA+L2*(rowAL*sigmaA*sigmaL+miuA*miuL))/(B12*(DB22-
DB12)*(sigmaA^2+miuA^2))+A2*DB22/B12*(DB22-DB12);

piB22=(A2-piB12*B12)/B22;

%===============================================================%
%-----  THIS IS THE FINAL STAGE : STAGE 3 -------------------%
%===============================================================%
%----- THIS IS THE PRICE OF BOND 1 AT STAGE 3 ----------------%
%===============================================================%
B13=0;
for i=3:t1;
B13=CB1(i)/((1+yieldA1(3))^(i-1))+B12;
end;
B13;

%===============================================================%
%----- THIS IS THE PRICE OF BOND 2 AT STAGE 3 ----------------%
%===============================================================%

B23=0;
for i=3:t2;
B23=CB2(i)/((1+yieldA2(3))^(i-1))+B22;
end;
B23;

%===============================================================%
%----- THIS IS THE DURATION OF BOND 1 AT STAGE 3 ------------%
%===============================================================%

DB13=0;
for i=3:t1;
DB13=CB1(i)*(i-1)/((1+yieldA1(3))^(i-1))+DB12;
end;
DB13=DB13/B13;

%===============================================================%
%----- THIS IS THE DURATION OF BOND 2 AT STAGE 3 ------------%
%===============================================================%
```

```
DB23=0;
for i=3:t2;
DB23=CB2(i)*(i-1)/((1+yieldA2(3))^(i-1))+DB22;
end;
DB23=DB23/B23;

%=============================================================%
%----- THIS IS THE ASSETS AMOUNT AT STAGE 3 -----------------%
%=============================================================%

A3=piB12*B13+piB12*CB1(3)+piB22*B22+piB22*CB2(3);

%=============================================================%
%----- THIS IS THE LIABILITY AMOUNT AT STAGE 3 --------------%
%=============================================================%

L3=CL*(1+0.13)^2;

%=============================================================%
%-----  THIS IS THE WEALTH AT STAGE 3 ----------------------%
%=============================================================%

display('THIS IS THE WEALTH AT STAGE 3:')
W3=A3-L3*(1+0.13)^2

%=============================================================%
%----- IF WE INVEST OUR MONEY IN THE RISK FREE MARKET, ------%
%----- WE GET...--------------------------------------------%
%=============================================================%

display('If you put your money in the risk-free market, your w
ealth will be:')

CA*(1+ratea(1))*(1+ratea(2))-CL


display('If you invest your money using the old method - durat
ion re-matching method, you will get:')

dmatching;
```

```
%================================================================%
%===== FILE: dmatching.m =====================================%
%================================================================%
load binfo;
load rinfo;
load stage1;

syms piB11 B11 B21 DB11 DB21;

%================================================================%
%----- B11 IS THE BOND PRICE OF BOND 1 AT STAGE 1 -----------%
%================================================================%

B11=0;
for i=1:t1;
B11=CB1(i)/((1+yieldA1(1))^(i-1))+B11;
end;
B11;

%================================================================%
%----- B21 IS THE BOND PRICE OF BOND 2 AT STAGE 1 -----------%
%================================================================%

B21=0;
for i=1:t2;
B21=CB2(i)/((1+yieldA2(1))^(i-1))+B21;
end;
B21;

%================================================================%
%----- DB11 IS THE DURATION OF BOND 1 AT STAGE 1 ------------%
%================================================================%

DB11=0;
for i=1:t1;
DB11=CB1(i)*(i-1)/((1+yieldA1(1))^(i-1))+DB11;
end;
DB11=DB11/B11;

%================================================================%
```

```
%----- DB21 IS THE DURATION OF BOND 2 AT STAGE 1 -----------%
%===========================================================%

DB21=0;
for i=1:t2;
DB21=CB2(i)*(i-1)/((1+yieldA2(1))^(i-1))+DB21;
DB21=DB21/B21;
end;

syms piB21;

piB11=(110000000-piB21*B21)/B11;
piB21=solve(DB11*B11*piB11+DB21*B21*piB21-3*110000000,piB21);

piB21=numeric(piB21);
piB11=(110000000-piB21*B21)/B11;

%===========================================================%
%----- STAGE 2 ---------------------------------------------%
%===========================================================%
%===========================================================%
%----- PRICE OF BOND 1 AT STAGE 2 --------------------------%
%===========================================================%

B12=0;
for i=2:t1;
B12=CB1(i)/((1+yieldA1(2))^(i-1))+B12;
end;
B12;

%===========================================================%
%----- PRICE OF BOND 2 AT STAGE 2 --------------------------%
%===========================================================%

B22=0;
for i=2:t2;
B22=CB2(i)/((1+yieldA2(2))^(i-1))+B22;
end;
B22;

%===========================================================%
```

```
%----- DURATION OF BOND 1 AT STAGE 2 -----------------------%
%==========================================================%

DB12=0;
for i=2:t1;
DB12=CB1(i)*(i-1)/((1+yieldA1(2))^(i-1))+DB12;
end;
DB12=DB12/B12;

%==========================================================%
%----- DURATION OF BOND 2 AT STAGE 2 -----------------------%
%==========================================================%

DB22=0;
for i=2:t2;
DB22=CB2(i)*(i-1)/((1+yieldA2(2))^(i-1))+DB22;
end;
DB22=DB22/B22;

A2=piB11*B12+piB21*B22+piB11*CB1(2)+piB21*CB2(2)+(CA-piB11*B11-
piB21*B21)*(1+ratea(1));

L2=CL*(1+ratel(1));

W2=A2-L2;

syms piB22;

piB12=(-piB22*B22+CL*(1+ratel(1)))/B12;
piB22=solve(piB12*B12*DB12+piB22*B22*DB22-2*CL,piB22);

numeric(piB22);
piB12=(-piB22*B22+CL*(1+ratel(1)))/B12;

%==========================================================%
%-----  THIS IS THE FINAL STAGE : STAGE 3 ------------------%
%==========================================================%
%==========================================================%
%----- THIS IS THE PRICE OF BOND 1 AT STAGE 3 --------------%
%==========================================================%
```

```
B13=0;
for i=3:t1;
B13=CB1(i)/((1+yieldA1(3))^(i-1))+B12;
end;
B13;


%==============================================================%
%----- THIS IS THE PRICE OF BOND 2 AT STAGE 3 ---------------%
%==============================================================%

B23=0;
for i=3:t2;
B23=CB2(i)/((1+yieldA2(3))^(i-1))+B22;
end;
B23;


%==============================================================%
%----- THIS IS THE DURATION OF BOND 1 AT STAGE 3 ------------%
%==============================================================%

DB13=0;
for i=3:t1;
DB13=CB1(i)*(i-1)/((1+yieldA1(3))^(i-1))+DB12;
end;
DB13=DB13/B13;

%==============================================================%
%----- THIS IS THE DURATION OF BOND 2 AT STAGE 3 ------------%
%==============================================================%

DB23=0;
for i=3:t2;
DB23=CB2(i)*(i-1)/((1+yieldA2(3))^(i-1))+DB22;
end;
DB23=DB23/B23;

A3=piB12*B13+piB22*B23+piB12*CB1(3)+piB22*CB2(3)+(A2-piB11*B11-piB21*B21)*(1+r
L3=CL*(1+ratel(1))*(1+ratel(2));
```

```
W3=numeric(A3-L3)


%==============================================================%
%===== FILE: binfo.m ==========================================%
%==============================================================%

%==============================================================%
%----- The Time period is from 1 to max([t1,t2,t3]) ---------%
%==============================================================%
%----- THE MATURITY OF THE BOND ONE ---------------------------%
%==============================================================%

t1=4;

%==============================================================%
%----- THE CASH FLOW OF THE BOND ONE --------------------------%
%==============================================================%

CB1(1)=0;
CB1(2)=50;
CB1(3)=100;
CB1(4)=0;
%==============================================================%
%----- THE METURITY OF THE BOND TWO ---------------------------%
%==============================================================%

t2=4;

%==============================================================%
%----- THE CASH FLOW OF BOND TWO ------------------------------%
%==============================================================%

CB2(1)=0;
CB2(2)=50;
CB2(3)=50;
CB2(4)=100;

%==============================================================%
%----- THE CAPITAL VALUE --------------------------------------%
```

```
%==============================================================%

CA=120000000;
%==============================================================%
%----- THE LIABILITY PERIOD -----------------------------------%
%==============================================================%

t3=2;

%==============================================================%
%----- THE LIABILITY CASH FLOW --------------------------------%
%==============================================================%

CL=110000000;

tmax=max([t1;t2;t3]);

save binfo CB1 CB2 t1 t2 t3 CA CL tmax
```